Creative Blocking

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1 Introduction

1.1 Definition. A *polygon* is a finite ordered list of points in the plane. The *edges* of a polygon are the directed line segments between consecutive points, including the edge from the last point back to the first. A polygon is *trivial* if it does not contain at least two distinct points.

In general, a polygon may have repeated points. Thus, edges with length zero or even repeated edges might occur. Furthermore, the shape that a polygon represents could be non-convex or even self-intersecting.

1.2 Definition. Let $\mathscr{P}$ be the set of non-trivial polygons, and let $T : \mathscr{P} \to \mathscr{P}$ be defined as follows. Given a polygon p, a square is erected on the right side of each edge of p, where "right" is defined as if one were walking on the edges of p in the order of the points. With respect to the square erected on an edge e, the side opposite of e is taken to be an edge of $T(p)$ with the same orientation as e. The endpoints of these edges are then connected to form $T(p)$. See Figure 1.

This definition leads to complicated and interesting shapes. See Figure 2. The construction of $T(p)$ from p can be clarified by identifying the edges of the two polygons with complex numbers. If $u, v \in \mathbb{C}$ are two consecutive edges of p, then they are also edges in $T(p)$, but with the new edge $i(u - v)$ between them. Recall that multiplication by i corresponds to a rotation counter-clockwise by $\pi/2$. See Figure 3.

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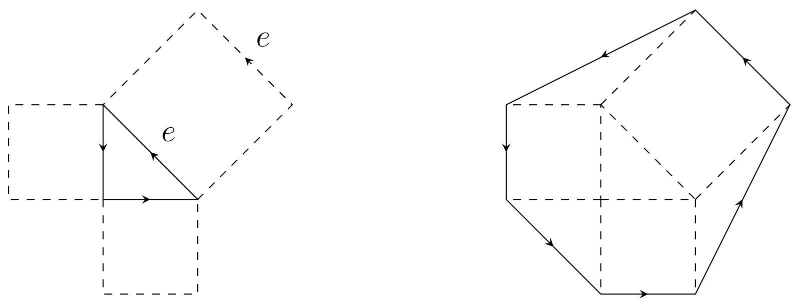

Figure 1: The construction of $T(p)$ from p, where p is an isosceles right triangle oriented counter-clockwise.

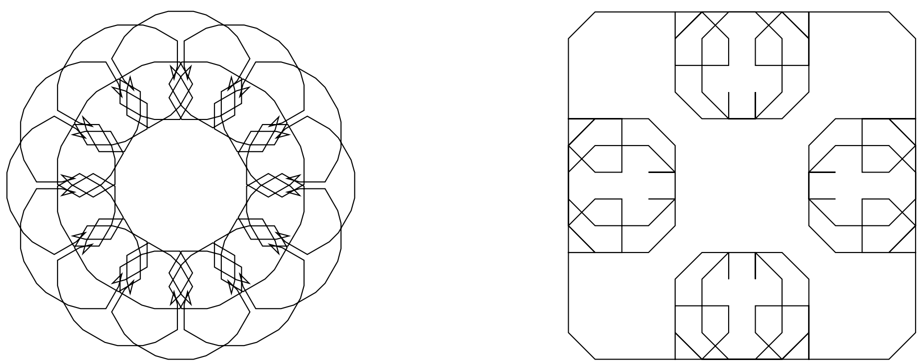

Figure 2: $T^6(p)$ and $T^6(q)$, where p and q respectively are a regular hexagon and a square oriented counter-clockwise.

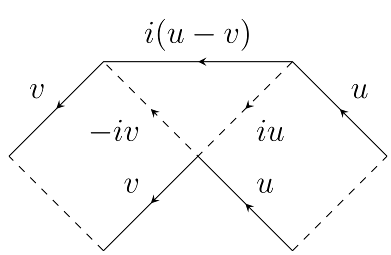

Figure 3: The edge $i(u-v)$ of $T(p)$ is created from the edges u and v of p. Recall that multiplication by i corresponds to a rotation counter-clockwise by $\pi/2$.

2 Growth rate of the perimeter

Given $p \in \mathscr{P}$, consider the sequence of polygons $T^n(p)$. Since each polygon in the sequence has twice as many edges as the previous one, we might expect the perimeter to grow like 2^n. This suggests the following definition.

2.1 Definition. For any polygon p, let $|p|$ denote its *perimeter* as given by the sum of the lengths of its edges. For $p \in \mathscr{P}$, let $G(p)$ denote its *growth rate* as given by

$$G(p) = \frac{1}{|p|} \lim_{n\to\infty} \frac{|T^n(p)|}{2^n}.$$

The normalizing factor $1/|p|$ is included so that the growth rate depends only on the shape of the polygon and not its size. To see how the growth rate can be calculated from the first few values of $|T^n(p)|$, we use the following lemma.

2.2 Lemma. For all non-negative integers n and all $p \in \mathscr{P}$,

$$4|T^n(p)| - |T^{n+2}(p)| + 2|T^{n+3}(p)| - |T^{n+4}(p)| = 0.$$

Proof. This is a specific instance of Theorem 4.1. $\square$

2.3 Theorem. For every $p \in \mathscr{P}$,

$$G(p) = \frac{1}{|p|} \cdot \frac{2|p| + |T(p)| + |T^3(p)|}{12}.$$

Proof. Lemma 2.2 gives a linear recurrence with characteristic equation $\lambda^4 - 2\lambda^3 + \lambda^2 - 4 = 0$. This equation has roots $\lambda_1 = 2$, $\lambda_2 = -1$, and $\lambda_{3,4} = \frac{1}{2}(1 \pm i\sqrt{7})$, so the solutions to the recurrence have the form

$$|T^n(p)| = \sum_{j=1}^{4} c_j \lambda_j^n,$$

where the constants c_j depend on p. Therefore,

$$G(p) = \frac{1}{|p|} \lim_{n\to\infty} \frac{1}{2^n} \sum_{j=1}^{4} c_j \lambda_j^n = \frac{c_1}{|p|},$$

using the facts that $|\lambda_1| = 2$ and $|\lambda_{2,3,4}| < 2$. The proposition follows by solving for c_1 in terms of the first few values of $|T^n(p)|$. $\square$

In the corollaries that follow, this formula is used to calculate the growth rates of triangles and regular polygons.

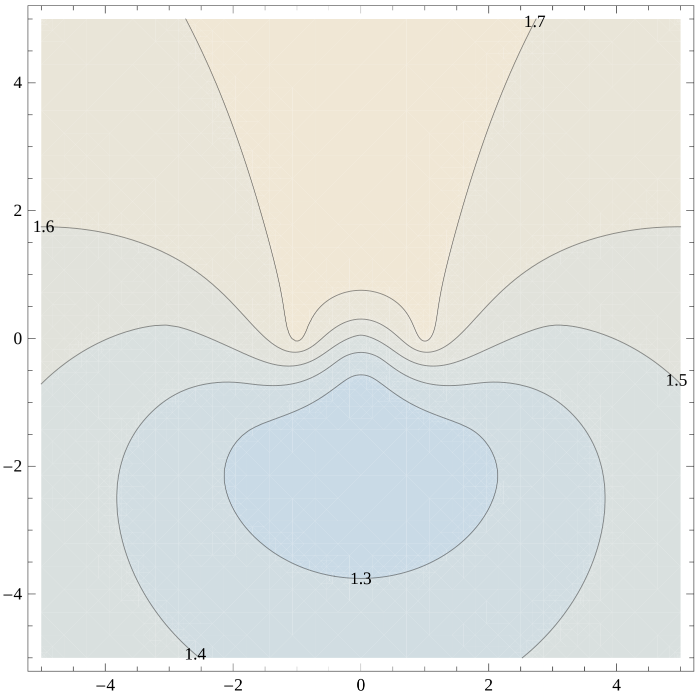

Figure 4: Contour plot of $G_\triangle(x, y)$. Note that the value may vary within a region.

2.1 Triangles

To systematically study the growth rates of triangles, we fix two vertices and vary the position of the third. Explicitly, we let $G_\triangle(x, y) = G((-1, 0), (x, y), (1, 0))$. The growth rate of any triangle can be computed from this by translating, rotating, and scaling so that the longest edge is from $(1, 0)$ to $(-1, 0)$. In this way, we see that each unique value of $G_\triangle(x, y)$ can be achieved with (x, y) in the union of the two disks of radius 2 centered at $(-1, 0)$ and $(1, 0)$.

2.4 Corollary. The exact value of $G_\triangle(x, y)$ is the following. See Figure 4.

$$\begin{aligned}
G_\triangle(x,y) = \Big(4 &+ 2\sqrt{x^2+y^2} + \sqrt{2}\sqrt{x^2+(y+1)^2} \\
&+ 2\sqrt{(x+1)^2+y^2} && + 2\sqrt{(x-1)^2+y^2} \\
&+ \sqrt{(x+3)^2+y^2} && + \sqrt{(x-3)^2+y^2} \\
&+ \sqrt{2}\sqrt{5+x(x+4)+y(y+2)} && + \sqrt{2}\sqrt{5+x(x-4)+y(y+2)} \\
&+ \sqrt{13+x(x+6)+y(y+4)} && + \sqrt{13+x(x-6)+y(y+4)} \\
&+ \sqrt{5+x(x+2)+y(y+4)} && + \sqrt{5+x(x-2)+y(y+4)} \\
&+ \sqrt{1+x(5x+2)+y(5y+4)} && + \sqrt{1+x(5x-2)+y(5y+4)}\Big) \\
&\div \Big(6\Big(2+\sqrt{(x-1)^2+y^2} && + \sqrt{(x+1)^2+y^2}\Big)\Big)
\end{aligned}$$

2.5 Corollary. For triangles, the growth rate is minimized at equilateral triangles oriented counter-clockwise and maximized at equilateral triangles oriented clockwise, with the following values.

$$
\begin{aligned}
G_\triangle(0,-\sqrt{3}) &= \tfrac{1}{12}(4-\sqrt{2}+2\sqrt{3}+\sqrt{6}+4\sqrt{4-\sqrt{3}}) && \approx 1.210 \\
G_\triangle(0,+\sqrt{3}) &= \tfrac{1}{36}(12+\sqrt{2}+6\sqrt{3}+\sqrt{6}+4\sqrt{2+\sqrt{3}}+12\sqrt{4+\sqrt{3}}) && \approx 1.742
\end{aligned}
$$

2.6 Corollary. For degenerate triangles, the growth rate is minimized when the three points are equally spaced and maximized when two of the points coincide, with the following values.

$$
\begin{aligned}
G_\triangle(\tfrac{1}{2},0) &= \tfrac{1}{24}(16+\sqrt{2}+2\sqrt{5}+2\sqrt{10}+2\sqrt{13}) && \approx 1.476 \\
G_\triangle(\pm 1,0) &= \tfrac{1}{12}(12+3\sqrt{2}+2\sqrt{5}) && \approx 1.726
\end{aligned}
$$

Note that $G_\triangle(x,y)$ approaches this second value whenever $\sqrt{x^2+y^2}\to\infty$. That is, whenever the distance from the origin to (x,y) tends to ∞.

2.2 Regular polygons

Let p_n and $\tilde{p}_n$ denote a regular n-gon oriented counter-clockwise and clockwise respectively. Some exact values of growth rates can be computed, but the minimum, maximum, and limiting values are only conjectured at this point. See Figure 5.

2.7 Corollary. For example,

$$
\begin{aligned}
G(p_2) &= \tfrac{1}{6}(4+\sqrt{2}+2\sqrt{5}) && \approx 1.648 \\
G(p_3) &= \tfrac{1}{12}(4-\sqrt{2}+2\sqrt{3}+\sqrt{6}+4\sqrt{4-\sqrt{3}}) && \approx 1.210 \\
G(p_4) &= \tfrac{1}{6}(4+\sqrt{2}) && \approx 0.902 \\
G(p_6) &= \tfrac{1}{4}(2-\sqrt{2}+\sqrt{6}) && \approx 0.759 \\
G(p_8) &= \tfrac{1}{3}(\sqrt{2}+\sqrt{2-\sqrt{2}}) && \approx 0.727
\end{aligned}
$$

2.8 Conjecture. The sequence $(G(p_n))$ decreases until reaching a minimum value when $n=7$, after which it is monotone increasing, while the sequence $(G(\tilde{p}_n))$ increases between $n=2$ and $n=3$, after which it is monotone decreasing. Furthermore, the two sequences converge to the same value,

$$
L = \lim_{n\to\infty} G(p_n) = \lim_{n\to\infty} G(\tilde{p}_n) = \tfrac{1}{6}(4+\sqrt{2}).
$$

For $n = 1\,000\,000$, the errors are the following.

$$
\begin{aligned}
G(p_n) - L &\approx -1.78768\times 10^{-6} \\
G(\tilde{p}_n) - L &\approx +3.88207\times 10^{-6}
\end{aligned}
$$

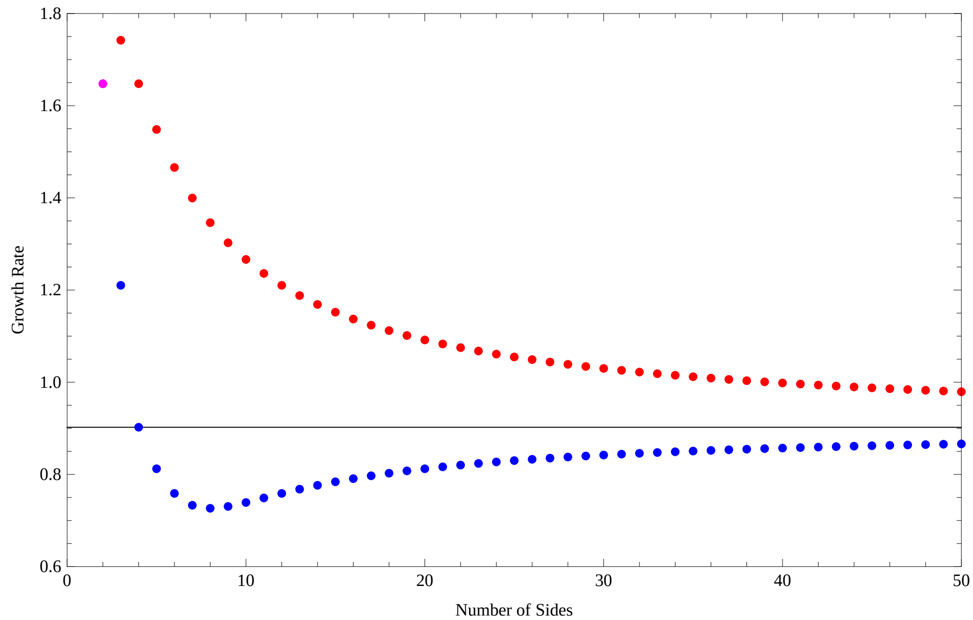

Figure 5: The values of $G(p_n)$ are shown in blue and $G(\tilde{p}_n)$ in red for $n \in \{3, \ldots, 50\}$. The values of $G(p_2)$ and $G(\tilde{p}_2)$ are equal and shown in purple. The conjectured limiting value is shown as a black line.

We can state with certainty the effect on p_n due to a single application of T.

2.9 Proposition. For all integers $n \geq 2$,

$$|T(p_n)| = \left(1 + \sqrt{2 - 2\cos(2\pi/n)}\right) \cdot |p_n|$$

Proof. Let x denote the side length of p_n, so that $|p_n| = nx$. The edges of $T(p_n)$ are another n edges of length x together with n other edges of length y, so $|T(p_n)| = nx + ny$. By the law of cosines, $y = x\sqrt{2 - 2\cos\theta}$, where θ is the interior angle of p_n. Therefore,

$$\begin{aligned} |T(p_n)| &= nx \cdot \left(1 + \sqrt{2 - 2\cos(2\pi/n)}\right) \\ &= |p_n| \cdot \left(1 + \sqrt{2 - 2\cos(2\pi/n)}\right). \end{aligned}$$

□

2.10 Corollary.

$$\lim_{n \to \infty} \frac{|T(p_n)|}{|p_n|} = 1$$

2.3 Extreme growth rates

2.11 Conjecture. The maximum growth rate for any polygon is $\frac{1}{3}(4 + \sqrt{2}) \approx 1.805$, and this occurs if and only if each point in the polygon is repeated at least once in a row. For example, (a, a, b, b) where $a = (0, 0)$ and $b = (1, 0)$.

The contribution to the growth rate due to an edge with repeated endpoints is independent of any other edges, so the growth rate is the same for all polygons with repeated points. It seems that the contribution to the growth rate due to a pair of edges increases with the cosine of the angle between them. With repeated points, all these angles are zero.

2.12 Conjecture. Since G is always non-negative, it has an infimum, which we conjecture is positive. To test this, we planned to generate random polygons and record those with the smallest growth rates.

3 Sum-of-squares perimeter

By removing the square roots from our concept of perimeter, nicer results are obtained.

3.1 Definition. For any polygon p, let $\|p\|$ denote its *square perimeter* as given by the sum of the squares of the lengths of its edges. For $p \in \mathscr{P}$, let $\Gamma(p)$ denote its *square growth rate* as given by

$$\Gamma(p) = \frac{1}{\|p\|} \lim_{n\to\infty} \frac{\|T^n(p)\|}{2^n}.$$

To see how the square growth rate can be calculated from the first few values of $\|T^n(p)\|$, we use a lemma identical in form to Lemma 2.2.

3.2 Lemma. For all non-negative integers n and all $p \in \mathscr{P}$,

$$4\|T^n(p)\| - \|T^{n+2}(p)\| + 2\|T^{n+3}(p)\| - \|T^{n+4}(p)\| = 0.$$

Proof. Like Lemma 2.2, this is a specific instance of Theorem 4.1. $\square$

3.3 Theorem. For every $p \in \mathscr{P}$,

$$\Gamma(p) = \frac{1}{\|p\|} \cdot \frac{2\|p\| + \|T(p)\| + \|T^3(p)\|}{12}.$$

Proof. The proof is identical in form to that of Theorem 2.3. $\square$

3.4 Conjecture. The solutions to the linear recurrence given in Lemma 3.2 that actually occur for some $p \in \mathscr{P}$ satisfy an additional constraint. Because the order of the linear recurrence is four, its characteristic equation has four roots λ_i, and its solutions depend on four constants c_i. See Theorem 2.3. In every case that was tested, it happened that $c_2 = 0$. This is equivalent to satisfying the third-order linear recurrence with roots $\lambda_{1,3,4}$:

$$4\|T^n(p)\| - 4\|T^{n+1}(p)\| + 3\|T^{n+2}(p)\| - \|T^{n+3}(p)\| = 0.$$

This would lead to the following improvement on Theorem 3.3:

$$\Gamma(p) = \frac{1}{\|p\|} \cdot \frac{2\|p\| - \|T(p)\| + \|T^2(p)\|}{4}.$$

As for the usual perimeter, the case $c_2 = 0$ was observed only for regular hexagons.

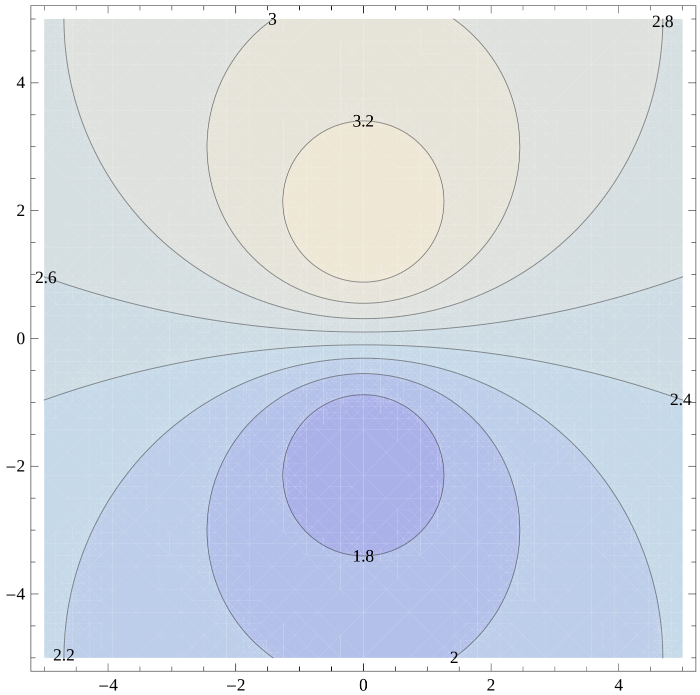

Figure 6: Contour plot of $\Gamma_\triangle(x, y)$. Note that the value may vary within a region.

3.1 Triangles

We restrict the input of Γ as we did for G, and define $\Gamma_\triangle(x, y) = \Gamma((-1, 0), (x, y), (1, 0))$.

3.5 Corollary.

$$\Gamma_\triangle(x, y) = \frac{5}{2} + \frac{3y}{3 + x^2 + y^2}$$

3.6 Corollary. For triangles, the square growth rate is minimized and maximized at the same places as the regular growth rate. That is, at equilateral triangles with counter-clockwise and clockwise orientation respectively. The actual values are the following.

$$\Gamma_\triangle(0, -\sqrt{3}) = \tfrac{5}{2} - \tfrac{\sqrt{3}}{2} \qquad \approx 1.634$$
$$\Gamma_\triangle(0, +\sqrt{3}) = \tfrac{5}{2} + \tfrac{\sqrt{3}}{2} \qquad \approx 3.366$$

3.7 Corollary. For degenerate triangles, the square growth rate is always the same: $5/2$. Also, $\Gamma_\triangle(x, y) \to 5/2$ whenever $\sqrt{x^2 + y^2} \to \infty$.

3.2 Regular polygons

Again, exact values of square growth rates can be computed, but the minimum, maximum, and limiting values are only conjectured. See Figure 7.

3.8 Corollary. For example,

$$\begin{aligned}
\Gamma(p_2) &= 3 && \approx 3.000 \\
\Gamma(p_3) &= \tfrac{1}{2}(5 - \sqrt{3}) && \approx 1.634 \\
\Gamma(p_4) &= 1 && \approx 1.000 \\
\Gamma(p_6) &= \tfrac{1}{2}(3 - \sqrt{3}) && \approx 0.634 \\
\Gamma(p_8) &= 2 - \sqrt{2} && \approx 0.586
\end{aligned}$$

3.9 Conjecture. The sequence $(\Gamma(p_n))$ decreases until reaching a minimum value when $n = 8$, after which it is monotone increasing, while the sequence $(\Gamma(\tilde{p}_n))$ increases between $n = 2$ and $n = 3$, after which it is monotone decreasing. Furthermore, the two sequences converge to the same value,

$$\lim_{n\to\infty} \Gamma(p_n) = \lim_{n\to\infty} \Gamma(\tilde{p}_n) = 1.$$

For $n = 1\,000\,000$, the errors are the following.

$$\begin{aligned}
\Gamma(p_n) - 1 &\approx -6.28317 \times 10^{-6} \\
\Gamma(\tilde{p}_n) - 1 &\approx +6.28321 \times 10^{-6}
\end{aligned}$$

Proposition 2.9 can be trivially modified as follows.

3.10 Proposition. For all integers $n \geq 2$,

$$\|T(p_n)\| = (3 - 2\cos 2\pi/n) \cdot \|p_n\|.$$

3.11 Corollary.

$$\lim_{n\to\infty} \frac{\|T(p_n)\|}{\|p_n\|} = 1$$

3.3 Extreme growth rates

3.12 Conjecture. We conjecture that there is a maximum and minimum square growth rate. However, the maximum square growth rate cannot occur under the same conditions as conjectured for the regular growth rate, because we have already seen several values higher than $\Gamma(a, a, b, b) = 2$, where $a = (0, 0)$ and $b = (1, 0)$. If d is a degenerate triangle, then $\Gamma(d) = 2.5$. Also, $\Gamma(a, b) = 3$, and $\Gamma(\tilde{p}_3) \approx 3.366$.

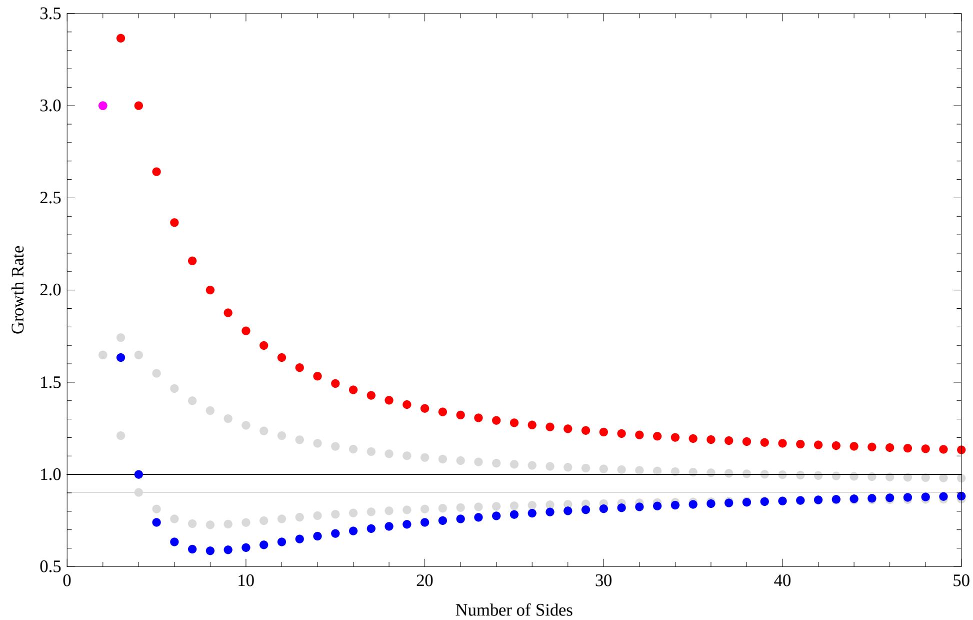

Figure 7: The values of $\Gamma(p_n)$ are shown in blue and $\Gamma(\tilde{p}_n)$ in red for $n \in \{3, \ldots, 50\}$. The values of $\Gamma(p_2)$ and $\Gamma(\tilde{p}_2)$ are equal and shown in purple. The conjectured limiting value is shown as a black line. The gray parts are from Figure 5 for comparison.

4 Main theorem

The sequence of perimeters $|T^n(p)|$ and the sequence of square perimeters $\|T^n(p)\|$ are generalized as a sequence A_n, depending on a function $f : \mathbb{C} \to \mathbb{R}$ corresponding respectively to the maps $z \mapsto |z|$ and $z \mapsto |z|^2$. It is shown that A_n satisfies the linear recurrence in Lemmas 2.2 and 3.2.

4.1 Theorem. Given a polygon $p \in \mathscr{P}$, let $e_{n,k}(p)$ denote the complex number with which we identify the k^{th} edge of $T^n(p)$. Suppose $f : \mathbb{C} \to \mathbb{R}$ satisfies $f(z) = f(iz)$ for all $z \in \mathbb{C}$, and let

$$A_n = \sum_k f(e_{n,k}(p)).$$

Then for all non-negative integers n,

$$4A_n - A_{n+2} + 2A_{n+3} - A_{n+4} = 0.$$

Proof. Let p, f, and A_n be given as above. Note that the equality for $n > 0$ follows from the case $n = 0$ by replacing p with $T^n(p)$; thus, we can assume $n = 0$ without loss of generality. In this setting, we only need to consider the five polygons p, $T(p)$, $T^2(p)$, $T^3(p)$, and $T^4(p)$.

As can be seen from Figure 3, the edges of $T(p)$ alternately depend on a single edge of p and two consecutive edges of p, and each pair of consecutive edges of $T(p)$ depends on only two consecutive edges of p. This dependence clearly extends to each edge of $T^2(p)$, $T^3(p)$, and $T^4(p)$, so we first examine only the edges determined by two consecutive edges $u, v \in \mathbb{C}$ of p.

Figure 3 shows that the edge $i(u-v)$ in $T(p)$ is created between the edges u and v. Applying T again will therefore create two new edges: $i(u-i(u-v))$ between u and $i(u-v)$, and $i(i(u-v)-v)$ between $i(u-v)$ and v. Iterating this process twice more leads to a total of seventeen edges of the form $au+bv$, which are shown in Table 1 after simplifying. Also shown in Table 1 are asterisks indicating which polygons contains which of these edges. Furthermore, each edge has a "type" m, assigned in such a way that edges of the same type differ only up to multiplication by a power of i.

The condition on f implies that $f(z) = f(iz) = f(-z) = f(-iz)$ for all $z \in \mathbb{C}$, so when f is applied to these seventeen edges, the output is determined by m. So, we record how many edges there are of each type in each polygon in Table 2, with the convention that the first and last edges are counted as one-half.

With this convention, the first five values of A_n can be computed as follows. Let $c_{m,n}$ be the (m,n)-entry in Table 2. For the k^{th} pair of consecutive edges of p, let $\varepsilon_{k,m}$ be an edge of type m. Then,

$$A_n = \sum_k \sum_{m=1}^{6} c_{m,n} \cdot f(\varepsilon_{k,m}),$$

where k varies over the pairs of consecutive edges of p. Since the second factor in the summands does not depend on n, we have

$$4A_0 - A_2 + 2A_3 - A_4 = \sum_k \sum_{m=1}^{6} \Big(4c_{m,0} - c_{m,2} + 2c_{m,3} - c_{m,4}\Big) \cdot f(\varepsilon_{k,m}).$$

The theorem follows from noting that each term in this sum is zero, which can be verified from Table 2. $\square$

Instead of looking at the edges of p, and then the edges of $T(p)$, and so on, we looked at the edges determined by a pair of consecutive edges of p, and then those determined by the next pair of consecutive edges of p, and so on. In this way, the collection of five polygons is divided into "sectors" in such a way that the recurrence can be verified for each sector individually. See Figure 8.

	$au+bv$	m
1	u	1
2	v	2
3	$u+iv$	3
4	$u+(-1+i)v$	4
5	$(1+i)u-v$	5
6	$iu-v$	3
7	$iu-(1+i)v$	4
8	$-iv$	2
9	$i(u-v)$	6
10	iu	1
11	$(-1+i)u-iv$	5
12	$-u-iv$	3
13	$-u+(1-i)v$	4
14	$-(1+i)u+v$	5
15	$-iu+v$	3
16	u	1
17	v	2

	p	$T(p)$	$T^2(p)$	$T^3(p)$	$T^4(p)$
1	$*$	$*$	$*$	$*$	$*$
2					$*$
3				$*$	$*$
4					$*$
5			$*$	$*$	$*$
6					$*$
7				$*$	$*$
8					$*$
9		$*$	$*$	$*$	$*$
10					$*$
11				$*$	$*$
12					$*$
13			$*$	$*$	$*$
14					$*$
15				$*$	$*$
16					$*$
17	$*$	$*$	$*$	$*$	$*$

Table 1: The consecutive edges $u, v \in \mathbb{C}$ of p lead to these seventeen edges $au+bv$ in $T^0(p), \ldots, T^4(p)$. A common value of m indicates that the two edges differ up to multiplication by a power of i. An asterisk indicates that the polygon contains the edge $au+bv$.

m	p	$T(p)$	$T^2(p)$	$T^3(p)$	$T^4(p)$
1	1/2	1/2	1/2	1/2	5/2
2	1/2	1/2	1/2	1/2	5/2
3	0	0	0	2	4
4	0	0	1	2	3
5	0	0	1	2	3
6	0	1	1	1	1

Table 2: For an edge e of type m, the coefficient of $f(e)$ in the defining sum of A_n that is contributed by the edges determined by the pair of consecutive edges u and v.

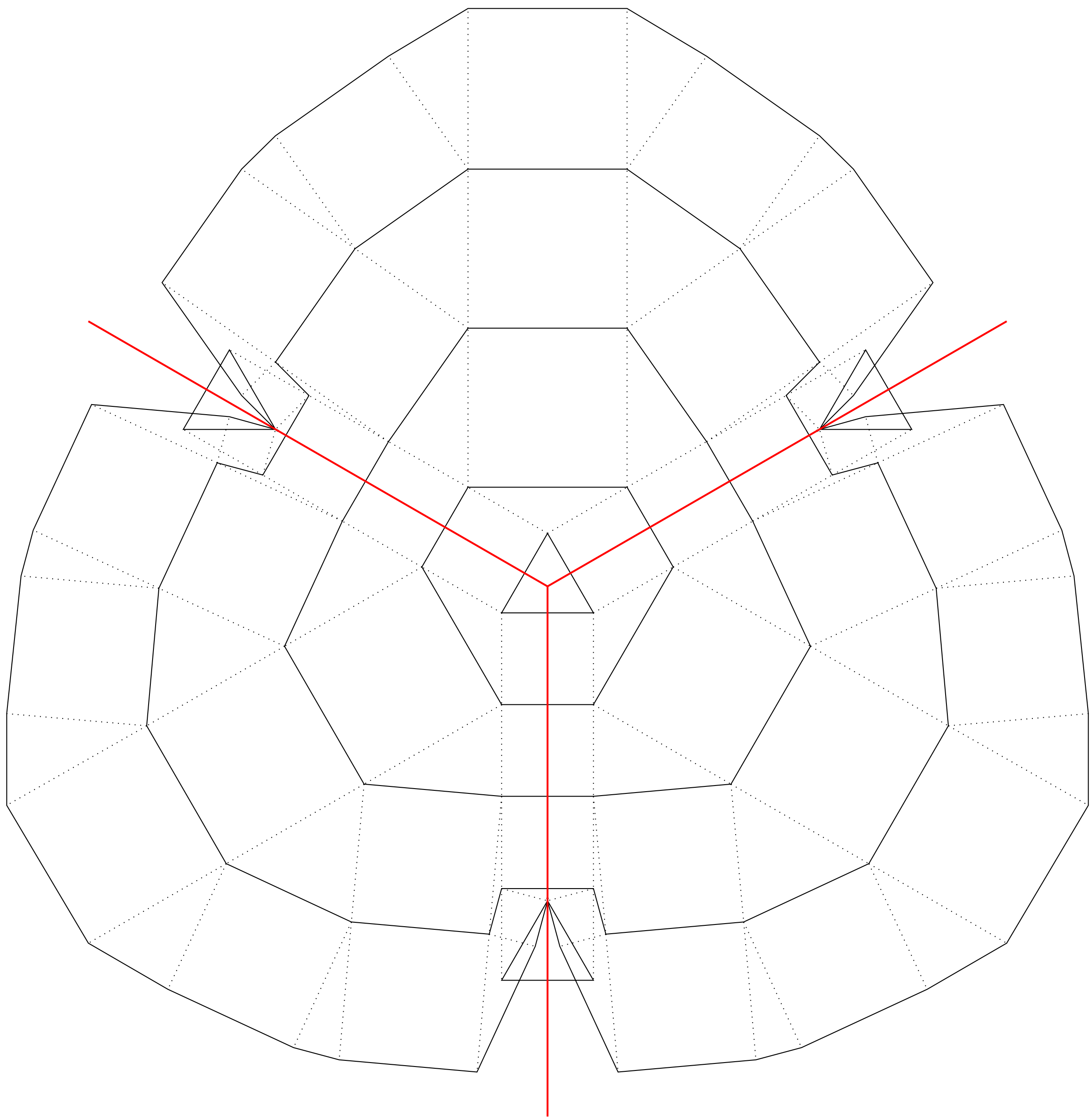

Figure 8: An equilateral triangle p oriented counter-clockwise, along with $T(p)$, $T^2(p)$, $T^3(p)$, and $T^4(p)$. The fourth iteration is self-intersecting. The dotted lines show the squares built in the process of applying T. There are three pairs of consecutive edges in p, one around each vertex, which divides the polygons into "sectors" as shown.