Appendix A

Errata*

The labeling is natural; for example, "4.15" means page 4, line 15, and "98.15–16" means page 98, lines 15–16. Snippets of the text are reproduced with the corrections (or their proximate locations) highlighted in yellow. The symbol $\ggg$ designates potentially confusing errata.

viii.22

> In Chapter 4 we state and prove exact formulas for the multiplier systems of $\eta(\tau)$ and $\vartheta(\tau)$.

ix.29

> There we apply Fejér's theorem on Fourier series and supply a reference to Titchmarsh.

ix.39

> (An exception is Chapter 8, which, as mentioned above, is taken from the unpublished manuscript of Atkin.)

4.15

> Since $(c', d') = 1$ we can determine integers a', b' with $a'd' - b'c' = 1$; that is,
>
> $$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma(1).$$

6.13

> If $1 \leqslant n \leqslant 4$, since $a \neq 0$, the closed intervals $[|a|(\sqrt{n}t - 1), |a|(\sqrt{n}t + 1)]$ with $t \in Z$, cover the entire real line.

10.3

> Then as a F.R. for Γ we may choose $\mathscr{R} = \bigcup_{i=1}^{\mu} A_i\{\mathscr{R}(\Gamma(1))\}$.

*These errata were compiled in 2016–2017 by Daniel Hirsbrunner, in consultation with Wladimir Pribitkin. Undoubtedly, the late Professor Marvin Knopp (who was aware of many of the typographical errors) would be extremely grateful to D. Hirsbrunner for his diligence. Please send all comments and suggestions to W. Pribitkin.

⋙ 10.20

Since A_i is actually a homeomorphism of $\mathscr{H}$ onto itself, it is easy to see that $A_i(\overline{\mathscr{R}(\Gamma(1))}) = \overline{A_i(\mathscr{R}(\Gamma(1)))} \subset \overline{\mathscr{R}}$.

10.22

As a F.R. for $\Gamma_0(p)$, p prime, we may choose

$$\mathscr{R}(\Gamma(1)) \cup \bigcup_{j=0}^{p-1} TS^j\{\mathscr{R}(\Gamma(1))\}.$$

14.12

Let $\mathscr{R}$ be a S.F.R. for Γ of the form $\mathscr{R} = \bigcup_{i=1}^{\mu} A_i\{\mathscr{R}(\Gamma(1))\}$.

⋙ 16.2

A matrix or linear fractional transformation $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \neq I$ such that $\alpha + \delta = \pm 2$ is called *parabolic*.

20.30

In virtue of the converse the existence of at most finitely many poles in $\overline{\mathscr{R}} \cap \mathscr{H}$ and the validity of the expansions (5) are equivalent conditions for functions meromorphic in $\mathscr{H}$, which satisfy (1) for the group Γ.

23.8

Then (10) takes the form

$$F(\tau) = K'\sigma(\tau) \sum_{n=-\infty}^{\infty} a'_n(j) e^{2\pi i(n+\kappa)A^{-1}\tau/\lambda}.$$

27.24

Thus for $\tau \in \mathscr{H}$,

$$\varphi(\tau) = |y^{r/2}F(\tau)| = \beta(c)(y')^{r/2} \left| \sum_{n+\kappa_j>0} a_n(j) e^{2\pi i(n+\kappa_j)(A_j^{-1}\tau)/\lambda_j} \right|$$

$$= \beta(c)(y')^{r/2} e^{-2\pi(n_0+\kappa_j)y'/\lambda_j} \left| \sum_{n \geqslant n_0} a_n(j) e^{2\pi i(n-n_0)A_j^{-1}\tau/\lambda_j} \right|,$$

where $n_0 + \kappa_j > 0$.

28.14

Then

$$\int_z^{z+\lambda} F(\zeta) e^{-2\pi i(n+\kappa)\zeta/\lambda} d\zeta$$
$$= \int_z^{z+\lambda} \left(\sum_{m+\kappa>0} a_m e^{2\pi i(m+\kappa)\zeta/\lambda} \right) e^{-2\pi i(n+\kappa)\zeta/\lambda} d\zeta$$
$$= \sum_{m+\kappa>0} \int_z^{z+\lambda} a_m e^{2\pi i(m-n)\zeta/\lambda} d\zeta = \lambda a_n,$$

where the integral is taken along the horizontal path.

30.6

We proceed as in the proof of Theorem 10 to find that if we consider the expansion of $F(\tau)$ at ∞,

$$F(\tau) = \sum_{m+\kappa \geqslant 0} a_m e^{2\pi i(m+\kappa)\tau/\lambda}, \qquad \operatorname{Im} \tau > 0,$$

we conclude that $|a_n| \leqslant C y^{-r/2} e^{2\pi(n+\kappa)y/\lambda}$ for $n = 0, 1, 2, \ldots$ and arbitrary $y > 0$.

⋙ 36.16

With these restrictions we may then interchange the order of summation to obtain

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{(1-x^2)\cdots(1-x^{2k})} \sum_{m=-\infty}^{\infty} x^{(m+k)^2} z^m$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k (xz^{-1})^k}{(1-x^2)\cdots(1-x^{2k})} \sum_{m=-\infty}^{\infty} x^{(m+k)^2} z^{m+k}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k (xz^{-1})^k}{(1-x^2)\cdots(1-x^{2k})} \sum_{m=-\infty}^{\infty} x^{m^2} z^m.$$

39.22

With

$$a_n = \int_0^1 \varphi(x) e^{2\pi i n x} dx,$$

we know that $\sum_{n=-\infty}^{\infty} a_n e^{-2\pi i n t}$ is Cesàro-summable to $\varphi(t)$, for all t.

39.23

This follows from Fejér's theorem, since $\varphi(t)$ is continuous for all t.

41.7

Therefore, by the identity theorem for analytic functions (11) holds for all complex z with $t > 0$.

43.17

In Theorem 9 we proved that $\eta(T\tau) = \eta(-1/\tau) = (-i)^{1/2}\tau^{1/2}\eta(\tau)$, while it is obvious from the definition of $\eta(\tau)$ that $\eta(S\tau) = \eta(\tau+1) = e^{\pi i/12}\eta(\tau)$.

52.8

But $v_\eta(I) = 1$ and the formula gives, in this case, $(\frac{0}{1})_* e^{\pi i(3-3)/12} = 1$.

54.5

Proceeding further in this same vein we obtain

$$\left(\frac{c}{c+d}\right)_* = \left(\frac{2}{|c+d|}\right)^\alpha \left(\frac{c_1}{|d|}\right)(-1)^{\frac{\operatorname{sign} c_1 - 1}{2}\frac{\operatorname{sign} d - 1}{2}}(-1)^{\frac{c_1-1}{2}\left(\frac{c}{2}+d-1\right)}$$
$$= \left(\frac{2}{|c+d|}\right)^\alpha \left(\frac{c_1}{d}\right)_* (-1)^{\frac{c_1-1}{2}\frac{c}{2}},$$

since d is odd.

⋙ 54.21

If b is odd, then $a \equiv -d \equiv \pm 1 \pmod 4$.

58.16

This is the same as

$$v_\eta(M')e^{\pi i/4}\exp\left\{\frac{-\pi i}{12}(2cd - 3cd + 3 + acd^2 - bdc^2)\right\}$$
$$= v_\eta(M')e^{\pi i/4}e^{-\pi i 3/12} = v_\eta(M'),$$

and the proof is complete for case 3(b).

60.3

Thus the proposed formula equals

$$v_\eta(M')(-1)^{\frac{|d|-1}{2}}(-1)^{\frac{\operatorname{sign} d-1}{2}}\exp\left\{\frac{-\pi i}{12}(2cd + 6d - 3 - 3cd + acd^2 - bdc^2)\right\}e^{\pi i/4}$$
$$= v_\eta(M')(-1)^{\frac{|d|-1}{2}}(-1)^{\frac{\operatorname{sign} d-1}{2}}\exp\left\{\frac{\pi i}{12}(3-6d)\right\}e^{\pi i/4}$$
$$= v_\eta(M')(-1)^{\frac{|d|-1}{2}}(-1)^{\frac{\operatorname{sign} d-1}{2}}e^{(\pi i/2)(1-d)} = v_\eta(M').$$

60.5

This completes the proof of case 3(c) and, with it, of Theorem 2.

61.16

As in case 1, the expression (4) holds for $v_\eta(M_1)^2$.

66.9

However, by Cauchy's theorem, $\int_{D_{N,\rho}} t^{\lambda-1}e^t dt$ is clearly *independent* of ρ, as long as $\rho > 0$.

82.16

Then using Lemma 1 of Chapter 4 we have

$$\left(\frac{c}{2c-h}\right) = \left(\frac{2}{2c-h}\right)^{\alpha}\left(\frac{c_1}{2c-h}\right) = (-1)^{\frac{(2c-h)^2-1}{8}\alpha}\left(\frac{2c-h}{c_1}\right)(-1)^{\frac{c_1-1}{2}\frac{2c-h-1}{2}}$$
$$= \cdots .$$

88.13

This method has been improved by Rademacher [*Proc. London Math. Soc., 43* (1937), pp. 241–254] to give an *exact formula* for $p(n)$, but we do not give a proof, since Rademacher's "improved circle method" is rather complicated, and since excellent expositions are already available [e.g., J. Lehner, *Discontinuous Groups and Automorphic Functions* (Providence: American Mathematical Society, 1964), pp. 302–313 and 350–351].

⋙ 89.4

The exact formula is

$$(2) \qquad p(n) = \frac{e^{\pi i/4}}{\sqrt{2}\pi}\sum_{k=1}^{\infty} A_k(n)k^{1/2}\frac{d}{dn}\left\{\frac{\sinh(\pi\sqrt{\frac{2}{3}}\sqrt{n-\frac{1}{24}}/k)}{\sqrt{n-\frac{1}{24}}}\right\},$$

for all n, where

$$A_k(n) = \sum_{\substack{h \,(\mathrm{mod}\ k)\\ (h,k)=1}} v_\eta(M_{k,-h})\exp\left\{2\pi i\frac{-(n-\frac{1}{24})h - \frac{1}{24}h'}{k}\right\}.$$

94.4

With $z' = x' + iy'$, this implies in particular, that $y' \geqslant \frac{\sqrt{3}}{2}$, $|x'| \leqslant \frac{1}{2}$.

94.11

But also, from the power-series expression for $\eta(z)^{-1}$,

$$|\eta(z')|^{-1} = |e^{-\pi i z'/12}|\left|\sum_{n=0}^{\infty} p(n)e^{2\pi i n z'}\right| \leqslant e^{\pi y'/12}\sum_{n=0}^{\infty} p(n)e^{-2\pi n y'}$$
$$= e^{\pi y'/12}\sum_{n=0}^{\infty} p(n)e^{-\pi n\sqrt{3}} = K_1 e^{\pi y'/12},$$

with K_1 a positive constant.

94.22

If $c = 0$, then $d = \pm 1$, and $y' = y = \varepsilon < \frac{1}{8}$, contradicting the fact that $y' \geqslant \frac{\sqrt{3}}{2}$.

⋙ 94.24

If $|d| \geqslant 1$, then $|cx + d| \geqslant |d| - |cx| \geqslant 1 - \frac{1}{2} = \frac{1}{2}$, so that

$$\frac{1}{y'} = \frac{(cx+d)^2}{y} + c^2 y \geqslant \frac{1}{4y} + y = \frac{1}{4\varepsilon} + \varepsilon > 2.$$

⋙ 95.1

Hence $y' < \frac{1}{2}$, a contradiction.

95.3

Hence $y' < \frac{1}{2}$, again contradicting the inequality $y' \geqslant \frac{\sqrt{3}}{2}$.

⋙ 96.18

We temporarily introduce the new convention $-\pi/2 \leqslant \arg z < \frac{3}{2}\pi$.

96.22

Recall that L_2 is the horizontal line segment from $-\sqrt{2\varepsilon} + i\varepsilon$ to $\sqrt{2\varepsilon} + i\varepsilon$, with $\varepsilon = (96m)^{-1/2} = \{96(n - \frac{1}{24})\}^{-1/2} < \frac{1}{8}$.

97.13†

$-\delta - i(1 - \delta^2)^{1/2}$

⋙ 98.15–16

Since $-\pi/2 \leqslant \arg z < \frac{3}{2}\pi$, we have $0 \leqslant \arg u < 2\pi$. With this substitution the integrand $-e^{-\pi i/4}e^{-2\pi i g(z)}\sqrt{z}dz$ becomes $e^{-2\pi m u - \pi/12u}\sqrt{u}du$.

†This is the label of the bottom-left point in Figure 6.

99.11, 99.13–14

We have

$$\int_{L_1^+(\delta)} e^{-2\pi m u - \pi/12u} \sqrt{u}\, du$$
$$= (1-\delta^2)^{1/2} \int_{L_1^+} e^{-2\pi m(\sqrt{1-\delta^2}u + i\delta)} \exp\left[-\frac{\pi}{12(\sqrt{1-\delta^2}u + i\delta)}\right]$$
$$\times \sqrt{(1-\delta^2)^{1/2}u + i\delta}\; du$$
$$\to \int_{L_1^+} e^{-2\pi m u - \pi/12u} \sqrt{u}\, du, \qquad \text{as } \delta \to 0_+,$$

by the bounded convergence theorem.

142.22

These congruences were given by Zuckerman [*Duke Math. J.* *5* (1939), pp. 88–110, esp. p. 89].

149.2

However, in this case the inequality $|c\omega_0 + d| \geqslant 1$ still holds.

⋙ 149.12–13

Page 45. There is an analogous result for $\eta^3(\tau)$: As a consequence of Corollary 6, we obtain

$$\eta^3(\tau) = e^{\pi i\tau/4} \sum_{m=0}^{\infty} (-1)^m (2m+1) e^{\pi i m(m+1)\tau}$$
$$= \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left(\frac{-1}{n}\right) n e^{2\pi i n^2 \tau/8},$$

where $\left(\frac{a}{b}\right)$ is Jacobi's symbol.

153.10

Fejér's theorem, 39

Appendix B

Corrigendum‡

B.1 Flaw in the Proof of Corollary 6 in Chapter 3

On page 38, lines 18–21, Knopp states,

> By Taylor's theorem,
> $$(1-\varepsilon)^m = 1 - m\varepsilon + \frac{m(m-1)}{2} t^{m-2}\varepsilon^2,$$
> where $1-\varepsilon \leqslant t \leqslant 1$, so that, for all integers m,
> $$|\rho| = \frac{m(m-1)}{2} t^{m-2}\varepsilon^2 < \tfrac{1}{2}(|m|+1)^2\varepsilon^2.$$

Note that $t \in (0,1]$, so $t^{m-2} \in (0,1]$ when $m-2 \geqslant 0$, and $t^{m-2} \in [1,\infty)$ otherwise. Thus, for $m \geqslant 2$, the bound on $|\rho|$ certainly does hold. It also holds for $m \in \{0,1\}$, since then $\rho = 0$. The problem lies in the negative integers m. If $m < 0$, then $m = -|m|$, and so the claimed bound on $|\rho|$ is equivalent to

$$\frac{-|m|(-|m|-1)}{2} t^{-|m|-2}\varepsilon^2 < \tfrac{1}{2}(|m|+1)^2\varepsilon^2,$$

or, after simplifying,

$$t^{-|m|-2} < \frac{|m|+1}{|m|}.$$

This inequality does hold, but only if t is sufficiently close to 1; or in other words, if ε is sufficiently small. So, for each negative integer m, there is a positive number $\varepsilon_{|m|}$ such that the given bound on $|\rho|$ is valid only if $\varepsilon < \varepsilon_{|m|}$. Now $(|m|+1)/|m| \to 1$, and therefore $\varepsilon_{|m|} \to 0$, as $|m| \to \infty$. So, strictly speaking, it is not possible to first apply the bound on $|\rho|$ for all integers m, and only afterwards let ε tend to 0. However, on page 39, lines 1–3, Knopp does exactly that, writing

> Now
> $$|R| \leqslant \varepsilon^{-1} \sum_{m=-\infty}^{\infty} |x|^{m(m+1)/2}|\rho| < \frac{\varepsilon}{2} \sum_{m=-\infty}^{\infty} (|m|+1)^2 |x|^{m(m+1)/2} = K\varepsilon,$$
> where K depends on x but not on ε. Thus for $|x| < 1$ $\lim_{\varepsilon\to 0+} R = 0$.

‡This corrigendum was composed in 2016–2017 by Daniel Hirsbrunner, in consultation with George Andrews and Wladimir Pribitkin. The late Professor Marvin Knopp (who was aware of the flaw in the proof) would be quite thankful for this correction. Please contact W. Pribitkin with any questions or comments.

What has actually been shown is that, for $\varepsilon < \varepsilon_{|M|}$, the following holds:

$$\begin{aligned}
\varepsilon^{-1} \sum_{m=-\infty}^{\infty} |x|^{m(m+1)/2}|\rho| &= \varepsilon^{-1} \sum_{m=-\infty}^{-|M|-1} |x|^{m(m+1)/2}|\rho| + \varepsilon^{-1} \sum_{m=-|M|}^{\infty} |x|^{m(m+1)/2}|\rho| \\
&< \varepsilon^{-1} \sum_{m=-\infty}^{-|M|-1} |x|^{m(m+1)/2}|\rho| + \frac{\varepsilon}{2} \sum_{m=-|M|}^{\infty} (|m|+1)^2|x|^{m(m+1)/2} \\
&\leqslant \varepsilon^{-1} \sum_{m=-\infty}^{-|M|-1} |x|^{m(m+1)/2}|\rho| + \frac{\varepsilon}{2} \sum_{m=-\infty}^{\infty} (|m|+1)^2|x|^{m(m+1)/2} \\
&= \varepsilon^{-1} \sum_{m=-\infty}^{-|M|-1} |x|^{m(m+1)/2}|\rho| + K\varepsilon.
\end{aligned}$$

Now, as $\varepsilon \to 0+$, the least negative integer M for which $\varepsilon < \varepsilon_{|M|}$ holds tends to $-\infty$. That is, the tail

$$\sum_{m=-\infty}^{-|M|-1} |x|^{m(m+1)/2}|\rho|$$

tends to 0; however, whether it does this faster than ε^{-1} tends to ∞ remains to be shown.

B.2 Corrected Proof of Corollary 6 in Chapter 3

The flaw noted above can be fixed easily by making the following changes suggested by W. Pribitkin.

38.4–5

> In Theorem 3 replace x by $x^{1/2}$ and z by $x^{1/2}(-1+\varepsilon)$, with $0 < \varepsilon < \frac{1}{2}$. For $|x| < 1$ and $0 < \varepsilon < \frac{1}{2}$, we get . . . .

38.21

> By Taylor's theorem,
>
> $$(1-\varepsilon)^m = 1 - m\varepsilon + \frac{m(m-1)}{2} t^{m-2}\varepsilon^2,$$
>
> where $1-\varepsilon \leqslant t \leqslant 1$, so that, for all integers m,
>
> $$|\rho| = \frac{m(m-1)}{2} t^{m-2}\varepsilon^2 < (|m|+1)^2 2^{|m|+1}\varepsilon^2.$$

39.2–3

> Now
>
> $$|R| \leqslant \varepsilon^{-1} \sum_{m=-\infty}^{\infty} |x|^{m(m+1)/2}|\rho| < 2\varepsilon \sum_{m=-\infty}^{\infty} (|m|+1)^2 2^{|m|}|x|^{m(m+1)/2} = L\varepsilon,$$
>
> where L depends on x but not on ε.

B.3 Alternative Proof of Corollary 6 in Chapter 3

The identity to be proved is, for $|x| < 1$,

$$\prod_{n=1}^{\infty}(1-x^n)^3 = \sum_{m=0}^{\infty}(-1)^m(2m+1)x^{m(m+1)/2}.$$

Knopp remarks on page 38, lines 1–3,

> One is tempted here simply to replace x by $x^{1/2}$ and z by $-x^{1/2}$ in Theorem 3, but unfortunately this reduces to 0 on both sides of the identity.

Theorem 3 in Chapter 3 states that, for suitable x and z,

$$\prod_{n=0}^{\infty}(1-x^{2n+2})(1+x^{2n+1}z)(1+x^{2n+1}z^{-1}) = \sum_{m=-\infty}^{\infty} x^{m^2}z^m.$$

Knopp's temptation works if both sides are first divided by $(1+xz^{-1})$, which removes a singularity. This modification was suggested by G. E. Andrews. The left-hand side of Theorem 3, after division by $(1+xz^{-1})$ and replacement of x by $x^{1/2}$ and z by $-x^{1/2}$, easily reduces to the left-hand side of Corollary 6. For the right-hand side, the infinite series can be folded over onto itself, and then a finite geometric series will appear. To wit,

$$\begin{aligned}
\sum_{m=-\infty}^{\infty} x^{m^2}z^m &= \sum_{m=0}^{\infty} x^{m^2}z^m + \sum_{m=-\infty}^{-1} x^{m^2}z^m \\
&= \sum_{m=0}^{\infty} x^{m^2}z^m + \sum_{m=0}^{\infty} x^{(-m-1)^2}z^{-m-1} \\
&= \sum_{m=0}^{\infty} (x^{m^2}z^m + x^{(m+1)^2}z^{-m-1}) \\
&= \sum_{m=0}^{\infty} x^{m^2}z^m(1+x^{2m+1}z^{-2m-1}) \\
&= \sum_{m=0}^{\infty} x^{m^2}z^m(1+(xz^{-1})^{2m+1}).
\end{aligned}$$

Dividing by $(1+xz^{-1})$, we find that this becomes

$$\begin{aligned}
\sum_{m=0}^{\infty} x^{m^2}z^m\left(\frac{1+(xz^{-1})^{2m+1}}{1+xz^{-1}}\right) &= \sum_{m=0}^{\infty} x^{m^2}z^m\left(\frac{1-(-xz^{-1})^{2m+1}}{1-(-xz^{-1})}\right) \\
&= \sum_{m=0}^{\infty} x^{m^2}z^m\sum_{j=0}^{2m}(-xz^{-1})^j,
\end{aligned}$$

and upon replacing x by $x^{1/2}$ and z by $-x^{1/2}$, we see that this becomes at last,

$$\begin{aligned}
\sum_{m=0}^{\infty}(x^{1/2})^{m^2}(-x^{1/2})^m\sum_{j=0}^{2m}(-x^{1/2}(-x^{1/2})^{-1})^j &= \sum_{m=0}^{\infty}(-1)^m x^{(m^2+m)/2}\sum_{j=0}^{2m}1^j \\
&= \sum_{m=0}^{\infty}(-1)^m(2m+1)x^{(m^2+m)/2}.
\end{aligned}$$