Infinite Product Generating Functions for Multidimensional Partitions

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Abstract

We present a proof of the infinite product representation of the generating function for ordinary partitions due to L. Euler in 1748 [6]; a partly new proof of the infinite product representation of the generating function for plane partitions due to P. A. MacMahon in 1916 [8]; a disproof of MacMahon's conjectured infinite product representation of the generating function for higher dimensional partitions due to A. O. L. Atkin, P. Bratley, I. G. Macdonald, and J. K. S. McKay in 1967 [2]; and a proof of the asymptotics of higher dimensional partitions due to D. P. Bhatia, M. A. Prasad, and D. Arora in 1997 [4], which matches with the asymptotics predicted by MacMahon's conjecture.

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1 Ordinary Partitions

An ordinary partition of n is a non-increasing list of positive integers whose sum is n; here, the number of ordinary partitions of n is denoted $p_1(n)$. For example, $p_1(4) = 5$ because there are five ordinary partitions of 4, namely (4), $(3,1)$, $(2,2)$, $(2,1,1)$, and $(1,1,1,1)$. The sequence of values taken on by $p_1(n)$, starting with $p_1(0) = 1$ for convenience, begins 1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, $\ldots$. The generating function of this sequence is defined as the formal power series $\sum_{n=0}^{\infty} p_1(n) q^n$. Questions of convergence will not be discussed here, except to say that q can be thought of as a complex variable with $|q| < 1$. There is an infinite product representation of this generating function, which appeared at least as early as L. Euler's 1748 book *Introductio in Analysin Infinitorum* [6].

1.1 Theorem. Formally,

$$\sum_{n=0}^{\infty} p_1(n) q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^m}.$$

Proof. Expand each factor of the infinite product into a geometric series:

$$\begin{aligned}
\prod_{m=1}^{\infty} \frac{1}{1-q^m} &= \frac{1}{1-q} \cdot \frac{1}{1-q^2} \cdot \frac{1}{1-q^3} \cdot \frac{1}{1-q^4} \cdot \cdots \\
&= \left(1 + q + q^2 + q^3 + q^4 + \cdots\right) \\
&\quad \cdot \left(1 + q^2 + q^{2\cdot 2} + q^{3\cdot 2} + q^{4\cdot 2} + \cdots\right) \\
&\quad \cdot \left(1 + q^3 + q^{2\cdot 3} + q^{3\cdot 3} + q^{4\cdot 3} + \cdots\right) \\
&\quad \cdot \left(1 + q^4 + q^{2\cdot 4} + q^{3\cdot 4} + q^{4\cdot 4} + \cdots\right) \cdot \cdots .
\end{aligned}$$

Now imagine multiplying out this infinite product. For example, one resulting term comes from taking q^2 from the first two factors and a 1 from all the remaining factors; thus, $q^2 \cdot q^2 = q^4$. In this way, each term that results from the expansion can be interpreted as an ordinary partition of some number. The term taken from the first factor gives the number of 1s in the ordinary partition; the term taken from the second factor gives the number of 2s in the ordinary partition; and so on. Thus, q^4 will appear precisely once for every ordinary partition of 4, and so its coefficient in the fully expanded series will be $p_1(4) = 5$. Note that $p_1(0)$ needs to be defined as 1 for the infinite product to give the right constant term. □

2 Plane Partitions

A plane partition of n is a two-dimensional array of positive integers whose sum is n, such that the array is non-increasing in both directions; here, the number of plane partitions of n is denoted $p_2(n)$. For example, $p_2(4) = 13$ because there are thirteen plane partitions of 4, namely

$$\begin{pmatrix} 4 & \\ & \end{pmatrix}, \quad \begin{pmatrix} 3 & \\ 1 & \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 \\ & \end{pmatrix}, \quad \begin{pmatrix} 2 & \\ 2 & \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 \\ & \end{pmatrix},$$

$$\begin{pmatrix} 2 & & \\ 1 & & \\ 1 & & \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & \\ 1 & & \\ & & \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 1 \\ & & \\ & & \end{pmatrix},$$

$$\begin{pmatrix} 1 & & & \\ 1 & & & \\ 1 & & & \\ 1 & & & \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & & \\ 1 & & & \\ 1 & & & \\ & & & \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & & \\ 1 & 1 & & \\ & & & \\ & & & \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & \\ 1 & & & \\ & & & \\ & & & \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ & & & \\ & & & \\ & & & \end{pmatrix}.$$

The sequence of values taken on by $p_2(n)$, starting with $p_2(0) = 1$ for convenience, begins 1, 1, 3, 6, 13, 24, 48, 86, 160, 282, 500, 859, 1479, ... The generating function of this sequence is defined as the formal power series $\sum_{n=0}^{\infty} p_2(n)q^n$, and there is also an infinite product representation of this generating function, which appeared at least as early as P. A. MacMahon's 1916 book *Combinatory Analysis* [8].

2.1 Theorem. Formally,

$$\sum_{n=0}^{\infty} p_2(n)q^n = \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m}.$$

The proof of this theorem is much more complicated than the previous one. We mostly follow the method presented in G. E. Andrews' 1976 book *The Theory of Partitions* [1]. The main difference is Lemma 2.10, where we present a new and more natural method suggested by G. E. Andrews. Also, the presentation of Lemmas 2.5 and 2.6 benefited from talks with A. J. Yee.

2.2 Definition. The q-Pochhammer symbol is defined as follows:

$$(a)_n = (a;q)_n = \prod_{j=1}^{n} (1 - aq^{j-1})$$

$$(a)_\infty = (a;q)_\infty = \prod_{j=1}^{\infty} (1 - aq^{j-1})$$

$$(a)_0 = (a;q)_0 = 1.$$

In turn, the q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k (q)_{n-k}}$$

when $0 \le m \le n$, and 0 otherwise.

2.3 Lemma. For all nonnegative integers n and m,

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m \end{bmatrix} + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}.$$

Proof. Expand the right-hand side and combine the resulting fractions:

$$\begin{aligned}
\begin{bmatrix} n-1 \\ m \end{bmatrix} + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} &= \frac{(q)_{n-1}}{(q)_m (q)_{n-m-1}} + q^{n-m} \frac{(q)_{n-1}}{(q)_{m-1}(q)_{n-m}} \\
&= \frac{(q)_{n-1}(1-q^{n-m}) + q^{n-m}(q)_{n-1}(1-q^m)}{(q)_m (q)_{n-m}} \\
&= \frac{(q)_{n-1}(1-q^{n-m}+q^{n-m}-q^n)}{(q)_m (q)_{n-m}} \\
&= \frac{(q)_{n-1}(1-q^n)}{(q)_m (q)_{n-m}} \\
&= \frac{(q)_n}{(q)_m (q)_{n-m}} \\
&= \begin{bmatrix} n \\ m \end{bmatrix}.
\end{aligned}$$

□

2.4 Lemma. For all nonnegative integers n and m,

$$\begin{bmatrix} n+m+1 \\ m+1 \end{bmatrix} = \sum_{j=0}^{n} \begin{bmatrix} m+j \\ m \end{bmatrix} q^j.$$

Proof. Induction on n. For the base case, simply note that if $n = 0$, then the equation reduces to $1 = 1$. For the inductive step, suppose that the equation holds for a specific value of n, and consider

$$\begin{bmatrix} n+m+2 \\ m+1 \end{bmatrix}.$$

Invoking Lemma 2.3 with n and m respectively replaced by $n+m+2$ and $m+1$, this can be expanded as

$$\begin{bmatrix} n+m+2 \\ m+1 \end{bmatrix} = \begin{bmatrix} n+m+1 \\ m+1 \end{bmatrix} + q^{n+1} \begin{bmatrix} n+m+1 \\ m \end{bmatrix}.$$

Now by the inductive hypothesis, the first term can be replaced with a sum like so:

$$\begin{bmatrix} n+m+2 \\ m+1 \end{bmatrix} = \sum_{j=0}^{n} \begin{bmatrix} m+j \\ m \end{bmatrix} q^j + q^{n+1} \begin{bmatrix} n+m+1 \\ m \end{bmatrix},$$

and conveniently, the extra term is precisely the one needed to complete the sum and prove the lemma. □

2.5 Lemma. For all nonnegative integers a, b, c, and d,

$$\sum_{j=d}^{b} \begin{bmatrix} a+j \\ c \end{bmatrix} q^j = q^{c-a} \left(\begin{bmatrix} a+b+1 \\ c+1 \end{bmatrix} - \begin{bmatrix} a+d \\ c+1 \end{bmatrix} \right).$$

Proof. Start with the left-hand side, and add and subtract c in the top entry of the q-binomial coefficient:

$$\sum_{j=d}^{b} \begin{bmatrix} a+j \\ c \end{bmatrix} q^j = \sum_{j=d}^{b} \begin{bmatrix} c+(a-c+j) \\ c \end{bmatrix} q^j.$$

Re-index with $k = a - c + j$:

$$= \sum_{k=a-c+d}^{a-c+b} \begin{bmatrix} c+k \\ c \end{bmatrix} q^{c-a+k}$$

$$= q^{c-a} \sum_{k=a-c+d}^{a-c+b} \begin{bmatrix} c+k \\ c \end{bmatrix} q^k.$$

Complete the sum so that it starts from $k = 0$:

$$= q^{c-a} \left(\sum_{k=0}^{a-c+b} \begin{bmatrix} c+k \\ c \end{bmatrix} q^k - \sum_{k=0}^{a-c+d-1} \begin{bmatrix} c+k \\ c \end{bmatrix} q^k \right).$$

Lastly, apply Lemma 2.4 to each sum:

$$= q^{c-a} \left(\begin{bmatrix} a+b+1 \\ c+1 \end{bmatrix} - \begin{bmatrix} a+d \\ c+1 \end{bmatrix} \right).$$

□

2.6 Lemma. Let $\pi_r(n_1,\ldots,n_k;q)$ denote the generating function for plane partitions with at most r columns, at most k rows, and with n_i the first entry in the ith row. Then,

$$\pi_r(n_1,\ldots,n_k;q) = q^{n_1+\cdots+n_k} \det\left(q^{(i-j)(i-j-1)/2}\begin{bmatrix} n_j+r-1 \\ r-i+j-1\end{bmatrix}\right)_{1\le i,j\le k}.$$

Proof. Induction on r. For the base case $r=1$, note that the q-binomial coefficient in the determinant is

$$\begin{bmatrix} n_j \\ j-i\end{bmatrix} = \begin{cases} 0 & \text{if } i<j \\ 1 & \text{if } i=j \\ \left[\begin{smallmatrix} n_j \\ j-i\end{smallmatrix}\right] & \text{if } i>j.\end{cases}$$

Thus, the matrix is upper triangular with 1s on the diagonal, and so its determinant is 1. Thus, the claim for the base case is that

$$\pi_1(n_1,\ldots,n_k;q) = q^{n_1+\cdots+n_k},$$

which is true because there is only one plane partition with at most 1 column and with that column fully specified.

For the inductive step, suppose that the claim holds for a particular $r \ge 1$. We need to express $\pi_{r+1}(n_1,\ldots,n_k;q)$ in terms of $\pi_r(m_1,\ldots,m_k;q)$. This can be done by separating out the contribution from the first column, and then summing over all possible second columns:

$$\pi_{r+1}(n_1,\ldots,n_k;q) = q^{n_1+\cdots+n_k}\sum_{m_k=0}^{n_k}\sum_{m_{k-1}=m_k}^{n_{k-1}}\cdots\sum_{m_1=m_2}^{n_1}\pi_r(m_1,\ldots,m_k;q).$$

Plugging in the inductive hypothesis to this gives

$$\begin{aligned}
&\pi_{r+1}(n_1,\ldots,n_k;q)\\
&= q^{n_1+\cdots+n_k}\sum_{m_k=0}^{n_k}\sum_{m_{k-1}=m_k}^{n_{k-1}}\cdots\sum_{m_1=m_2}^{n_1} q^{m_1+\cdots+m_k}\det\left(q^{(i-j)(i-j-1)/2}\begin{bmatrix} m_j+r-1 \\ r-i+j-1\end{bmatrix}\right)_{1\le i,j\le k}\\
&= q^{n_1+\cdots+n_k}\sum_{m_k=0}^{n_k} q^{m_k}\sum_{m_{k-1}=m_k}^{n_{k-1}} q^{m_{k-1}}\cdots\sum_{m_1=m_2}^{n_1} q^{m_1}\det\left(q^{\binom{i-j}{2}}\begin{bmatrix} m_j+r-1 \\ r-i+j-1\end{bmatrix}\right)_{1\le i,j\le k}.
\end{aligned}$$

For convenience, define

$$A = \left(q^{\binom{i-j}{2}}\begin{bmatrix} m_j+r-1 \\ r-i+j-1\end{bmatrix}\right)_{1\le i,j\le k}.$$

Now look at the innermost summation and expand the determinant along the first column:

$$\sum_{m_1=m_2}^{n_1} q^{m_1}\det(A) = \sum_{m_1=m_2}^{n_1} q^{m_1}\sum_{i=1}^{k} q^{\binom{i-1}{2}}\begin{bmatrix} m_1+r-1 \\ r-i\end{bmatrix} A^*_{i,1}.$$

Here, $A^*_{i,1}$ is the so-called cofactor; that is, $(-1)^{i+1}$ times the $(k-1)\times(k-1)$-matrix obtained from the original matrix by removing the i-th row and first column. Now interchange the order of summation to obtain:

$$\sum_{i=1}^{k} q^{\binom{i-1}{2}}\left(\sum_{m_1=m_2}^{n_1} q^{m_1}\begin{bmatrix} m_1+r-1 \\ r-i\end{bmatrix}\right) A^*_{i,1}.$$

By Lemma 2.5, the inner sum can be evaluated as follows:

$$\sum_{m_1=m_2}^{n_1} q^{m_1} \begin{bmatrix} m_1+r-1 \\ r-i \end{bmatrix} = q^{1-i} \left(\begin{bmatrix} n_1+r \\ r-i+1 \end{bmatrix} - \begin{bmatrix} m_2+r-1 \\ r-i+1 \end{bmatrix} \right).$$

Now we can un-expand the determinant along the first column and obtain

$$\pi_{r+1}(n_1,\ldots,n_k;q) = q^{n_1+\cdots+n_k} \sum_{m_k=0}^{n_k} q^{m_k} \sum_{m_{k-1}=m_k}^{n_{k-1}} q^{m_{k-1}} \cdots \sum_{m_2=m_3}^{n_2} q^{m_2} \det(B)$$

where B is the same as the original matrix

$$\left(q^{\binom{i-j}{2}} \begin{bmatrix} m_j+r-1 \\ r-i+j-1 \end{bmatrix} \right)_{1\le i,j\le k}$$

except the first column is now replaced with

$$q^{\binom{i-1}{2}} q^{1-i} \left(\begin{bmatrix} n_1+r \\ r-i+1 \end{bmatrix} - \begin{bmatrix} m_2+r-1 \\ r-i+1 \end{bmatrix} \right) = q^{(i-1)(i-4)/2} \left(\begin{bmatrix} n_1+r \\ r-i+1 \end{bmatrix} - \begin{bmatrix} m_2+r-1 \\ r-i+1 \end{bmatrix} \right).$$

To clean this up, we perform the following column operation, which of course does not affect the value of the determinant. Add q^{-1} times the second column to the first column. This cancels out the second term in the entries in the first column, leaving only

$$q^{(i-1)(i-4)/2} \begin{bmatrix} n_1+r \\ r-i+1 \end{bmatrix}.$$

Thus,

$$\pi_{r+1}(n_1,\ldots,n_k;q) = q^{n_1+\cdots+n_k} \sum_{m_k=0}^{n_k} q^{m_k} \sum_{m_{k-1}=m_k}^{n_{k-1}} q^{m_{k-1}} \cdots \sum_{m_2=m_3}^{n_2} q^{m_2} \det(C)$$

where

$$C_{i,j} = \begin{cases} q^{(i-1)(i-4)/2} \begin{bmatrix} n_1+r \\ r-i+1 \end{bmatrix} & \text{if } j=1 \\ \\ q^{\binom{i-j}{2}} \begin{bmatrix} m_j+r-1 \\ r-i+j-1 \end{bmatrix} & \text{if } j\ge 2. \end{cases}$$

Next, repeat the whole process for the new innermost summation. This time, expand $\det(C)$ along the second column, interchange the order of summation, apply Lemma 2.5, and then un-expand the determinant with a new matrix D:

$$\begin{aligned} \sum_{m_2=m_3}^{n_2} q^{m_2} \det(C) &= \sum_{m_2=m_3}^{n_2} q^{m_2} \sum_{i=1}^{k} q^{\binom{i-2}{2}} \begin{bmatrix} m_2+r-1 \\ r-i+1 \end{bmatrix} C^*_{i,2} \\ &= \sum_{i=1}^{k} q^{\binom{i-2}{2}} \left(\sum_{m_2=m_3}^{n_2} q^{m_2} \begin{bmatrix} m_2+r-1 \\ r-i+1 \end{bmatrix} \right) C^*_{i,2} \\ &= \sum_{i=1}^{k} q^{\binom{i-2}{2}} q^{2-i} \left(\begin{bmatrix} n_2+r \\ r-i+2 \end{bmatrix} - \begin{bmatrix} m_3+r-1 \\ r-i+2 \end{bmatrix} \right) C^*_{i,2} \\ &= \det(D), \end{aligned}$$

where D is the same as C except the second column is now replaced with

$$q^{\binom{i-2}{2}}q^{2-i}\left(\begin{bmatrix} n_2+r \\ r-i+2 \end{bmatrix} - \begin{bmatrix} m_3+r-1 \\ r-i+2 \end{bmatrix}\right) = q^{(i-2)(i-5)/2}\left(\begin{bmatrix} n_2+r \\ r-i+2 \end{bmatrix} - \begin{bmatrix} m_3+r-1 \\ r-i+2 \end{bmatrix}\right)$$

This time for clean up, add q^{-1} times the third column to the second column, leaving only

$$q^{(i-2)(i-5)/2}\begin{bmatrix} n_2+r \\ r-i+2 \end{bmatrix}.$$

Thus,

$$\pi_{r+1}(n_1,\ldots,n_k;q) = q^{n_1+\cdots+n_k}\sum_{m_k=0}^{n_k} q^{m_k}\sum_{m_{k-1}=m_k}^{n_{k-1}} q^{m_{k-1}}\cdots\sum_{m_3=m_4}^{n_3} q^{m_3}\det(E)$$

where

$$E_{i,j} = \begin{cases} q^{(i-j)(i-j-3)/2}\begin{bmatrix} n_j+r \\ r-i+j \end{bmatrix} & \text{if } j \le 2 \\ \\ q^{\binom{i-j}{2}}\begin{bmatrix} m_j+r-1 \\ r-i+j-1 \end{bmatrix} & \text{if } j \ge 3. \end{cases}$$

Repeating this process $k-3$ more times shows that

$$\pi_{r+1}(n_1,\ldots,n_k;q) = q^{n_1+\cdots+n_k}\sum_{m_k=0}^{n_k} q^{m_k}\det(X)$$

where

$$X_{i,j} = \begin{cases} q^{(i-j)(i-j-3)/2}\begin{bmatrix} n_j+r \\ r-i+j \end{bmatrix} & \text{if } j \le k-1 \\ \\ q^{\binom{i-j}{2}}\begin{bmatrix} m_j+r-1 \\ r-i+j-1 \end{bmatrix} & \text{if } j = k. \end{cases}$$

We repeat the process one last time, but this time it is slightly different because the summation begins at $m_k = 0$ rather than $m_k = m_{k+1}$. So, expand $\det(X)$ along the last column, interchange the order of summation, apply Lemma 2.5, and then un-expand the determinant with a new matrix Y:

$$\begin{aligned} \sum_{m_k=0}^{n_k} q^{m_k}\det(X) &= \sum_{m_k=0}^{n_k} q^{m_k}\sum_{i=1}^{k} q^{\binom{i-k}{2}}\begin{bmatrix} m_k+r-1 \\ r-i+k-1 \end{bmatrix} X^*_{i,k} \\ &= \sum_{i=1}^{k} q^{\binom{i-k}{2}}\left(\sum_{m_k=0}^{n_k} q^{m_k}\begin{bmatrix} m_k+r-1 \\ r-i+k-1 \end{bmatrix}\right) X^*_{i,k} \\ &= \sum_{i=1}^{k} q^{\binom{i-k}{2}}q^{k-i}\left(\begin{bmatrix} n_k+r \\ r-i+k \end{bmatrix} - \begin{bmatrix} r-1 \\ r-i+k \end{bmatrix}\right) X^*_{i,k} \\ &= \sum_{i=1}^{k} q^{\binom{i-k}{2}}q^{k-i}\begin{bmatrix} n_k+r \\ r-i+k \end{bmatrix} X^*_{i,k} \\ &= \sum_{i=1}^{k} q^{(i-k)(i-k-3)/2}\begin{bmatrix} n_k+r \\ r-i+k \end{bmatrix} X^*_{i,k} \\ &= \det(Y), \end{aligned}$$

where

$$Y_{i,j} = q^{(i-j)(i-j-3)/2} \begin{bmatrix} n_j + r \\ r - i + j \end{bmatrix}.$$

This time there is no clean up required because $\left[\begin{smallmatrix} r-1 \\ r-i+k \end{smallmatrix}\right] = 0$.

Thus,

$$\pi_{r+1}(n_1, \ldots, n_k; q) = q^{n_1+\cdots+n_k} \det(Y).$$

This is not quite the claim yet, though. We define a new matrix Z, which is obtained from Y by multiplying the ith row by q^i and dividing the jth column by q^j. Note that $\det(Z) = \det(Y)$, and the new matrix has entries

$$\begin{aligned} Z_{i,j} &= q^{(i-j)(i-j-3)/2+(i-j)} \begin{bmatrix} n_j + r \\ r - i + j \end{bmatrix} \\ &= q^{(i-j)(i-j-3)/2+2(i-j)/2} \begin{bmatrix} n_j + r \\ r - i + j \end{bmatrix} \\ &= q^{(i-j)(i-j-3+2)/2} \begin{bmatrix} n_j + r \\ r - i + j \end{bmatrix} \\ &= q^{(i-j)(i-j-1)/2} \begin{bmatrix} n_j + r \\ r - i + j \end{bmatrix}. \end{aligned}$$

Hence,

$$\pi_{r+1}(n_1, \ldots, n_k; q) = q^{n_1+\cdots+n_k} \det(Z),$$

and Z has the claimed form. □

2.7 Lemma. For all nonnegative integers B and t,

$$(A)_{B-t} = \frac{(A)_B}{(-1)^t A^t q^{Bt-\binom{t+1}{2}} (q^{1-B}/A)_t}.$$

Proof. Multiply and divide the left-hand side by $(Aq^{B-t})_t$ to complete the product:

$$\begin{aligned} (A)_{B-t} &= \frac{(A)_{B-t}(Aq^{B-t})_t}{(Aq^{B-t})_t} \\ &= \frac{(A)_B}{(Aq^{B-t})_t}. \end{aligned}$$

Now flip the product in the denominator:

$$\begin{aligned} &= \frac{(A)_B}{(1 - Aq^{B-t})(1 - Aq^{B-t+1}) \cdots (1 - Aq^{B-1})} \\ &= \frac{(A)_B}{(-Aq^{B-t})(1 - q^{t-B}/A)(-Aq^{B-t+1})(1 - q^{t-B-1}/A) \cdots (-Aq^{B-1})(1 - q^{1-B/A})} \\ &= \frac{(A)_B}{(-1)^t A^t q^{Bt-(1+2+\cdots+t)} (1 - q^{t-B}/A)(1 - q^{t-B-1}/A) \cdots (1 - q^{1-B}/A)} \\ &= \frac{(A)_B}{(-1)^t A^t q^{Bt-\binom{t+1}{2}} (q^{1-B}/A)_t}. \end{aligned}$$

□

2.8 Definition. We will encounter the following type of q-hypergeometric series:

$$ {}_3\phi_2\left(\begin{matrix} a_1 & a_2 & a_3 \\ & b_1 & b_2 \end{matrix}; q, z\right) = \sum_{t=0}^{\infty} \frac{(a_0;q)_t(a_1;q)_t(a_2;q)_t}{(q;q)_t(b_1;q)_t(b_2;q)_t} z^t. $$

2.9 Lemma. For all nonnegative integers n,

$$ {}_3\phi_2\left(\begin{matrix} q^{-n} & a & b \\ & c & q^{1-n}ab/c \end{matrix}; q, q\right) = \frac{(c/a)_n(c/b)_n}{(c)_n(c/ab)_n}. $$

This result is known as the q-Pfaff–Saalschutz summation.

Proof. Label the left-hand side $S_n(a,b,c)$ and the right-hand side as $T_n(a,b,c)$. First note that when $n=0$, both are clearly equal to 1. Now consider the difference $S_n(a,b,c) - S_{n-1}(a,b,c)$

$$ S_n(a,b,c) - S_{n-1}(a,b,c) = {}_3\phi_2\left(\begin{matrix} q^{-n} & a & b \\ & c & q^{1-n}ab/c \end{matrix}; q, q\right) - {}_3\phi_2\left(\begin{matrix} q^{1-n} & a & b \\ & c & q^{2-n}ab/c \end{matrix}; q, q\right) $$

$$ = \sum_{t=0}^{\infty} \frac{(q^{-n})_t(a)_t(b)_t}{(q)_t(c)_t(q^{1-n}ab/c)_t} q^t - \sum_{t=0}^{\infty} \frac{(q^{1-n})_t(a)_t(b)_t}{(q)_t(c)_t(q^{2-n}ab/c)_t} q^t $$

$$ = \sum_{t=1}^{\infty} \frac{(a)_t(b)_t}{(q)_t(c)_t} \left(\frac{(q^{-n})_t}{(q^{1-n}ab/c)_t} - \frac{(q^{1-n})_t}{(q^{2-n}ab/c)_t} \right) q^t $$

$$ = \sum_{t=1}^{\infty} \frac{(a)_t(b)_t}{(q)_t(c)_t} \left(\frac{(q^{-n})_t(1-q^{1-n+t}ab/c)}{(q^{1-n}ab/c)_{t+1}} - \frac{(q^{1-n})_t(1-q^{1-n}ab/c)}{(q^{1-n}ab/c)_{t+1}} \right) q^t $$

$$ = \sum_{t=1}^{\infty} \frac{(a)_t(b)_t}{(q)_t(c)_t} \left(\frac{(q^{1-n})_{t-1}\left((1-q^{-n})(1-q^{1-n+t}ab/c) - (1-q^{t-n})(1-q^{1-n}ab/c)\right)}{(q^{1-n}ab/c)_{t+1}} \right) q^t $$

$$ = \sum_{t=1}^{\infty} \frac{(a)_t(b)_t(q^{1-n})_{t-1}}{(q)_t(c)_t(q^{1-n}ab/c)_{t+1}} \left((1-q^{-n})(1-q^{1-n+t}ab/c) - (1-q^{t-n})(1-q^{1-n}ab/c)\right) q^t $$

$$ = \sum_{t=1}^{\infty} \frac{(a)_t(b)_t(q^{1-n})_{t-1}}{(q)_t(c)_t(q^{1-n}ab/c)_{t+1}} (1-q^t)(q^{1-n}ab/c - q^{-n}) q^t $$

$$ = \sum_{t=1}^{\infty} \frac{(a)_t(b)_t(q^{1-n})_{t-1}}{(q)_{t-1}(c)_t(q^{1-n}ab/c)_{t+1}} q^{-n}(qab/c - 1) q^t $$

Reindex so that the summation begins at $t=0$:

$$ = \sum_{t=0}^{\infty} \frac{(a)_{t+1}(b)_{t+1}(q^{1-n})_t}{(q)_t(c)_{t+1}(q^{1-n}ab/c)_{t+2}} q^{-n}(qab/c - 1) q^{t+1}, $$

and pull out a few factors to obtain a standard hypergeometric series:

$$= \frac{(1-a)(1-b)q^{-n+1}(qab/c-1)}{(1-c)(1-q^{1-n}ab/c)(1-q^{2-n}ab/c)} \sum_{t=0}^{\infty} \frac{(aq)_t(bq)_t(q^{1-n})_t}{(q)_t(cq)_t(q^{3-n}ab/c)_t} q^t$$

$$= \frac{(1-a)(1-b)q^{-n+1}(qab/c-1)}{(1-c)(1-q^{1-n}ab/c)(1-q^{2-n}ab/c)} {}_3\phi_2 \left(\begin{matrix} aq & bq & q^{-(n-1)} \\ & cq & q^{1-(n-1)}(aq)(bq)/(cq) \end{matrix} ; q, q \right)$$

$$= \frac{(1-a)(1-b)q^{-n+1}(qab/c-1)}{(1-c)(1-q^{1-n}ab/c)(1-q^{2-n}ab/c)} S_{n-1}(aq, bq, cq).$$

Thus, we have a recurrence relation that gives S_n in terms of S_{n-1}. Now consider $T_n(a,b,c) - T_{n-1}(a,b,c)$:

$$\begin{aligned}
T_n(a,b,c) - T_{n-1}(a,b,c) &= \frac{(c/a)_n(c/b)_n}{(c)_n(c/ab)_n} - \frac{(c/a)_{n-1}(c/b)_{n-1}}{(c)_{n-1}(c/ab)_{n-1}} \\
&= \frac{(c/a)_n(c/b)_n}{(c)_n(c/ab)_n} - \frac{(c/a)_{n-1}(c/b)_{n-1}(1-cq^{n-1})(1-q^{n-1}c/ab)}{(c)_n(c/ab)_n} \\
&= \frac{(c/a)_{n-1}(c/b)_{n-1}}{(c)_n(c/ab)_n} \Big((1-q^{n-1}c/a)(1-q^{n-1}c/b) - (1-cq^{n-1})(1-q^{n-1}c/ab) \Big) \\
&= \frac{(c/a)_{n-1}(c/b)_{n-1}}{(c)_n(c/ab)_n} cq^{n-1} \left(1 + \frac{1}{ab} - \frac{1}{a} - \frac{1}{b} \right) \\
&= \frac{(c/a)_{n-1}(c/b)_{n-1}}{(c)_n(c/ab)_n} cq^{n-1} \frac{(1-a)(1-b)}{ab} \\
&= \frac{(c/a)_{n-1}(c/b)_{n-1}}{(cq)_{n-1}(c/ab)_n} cq^{n-1} \frac{(1-a)(1-b)}{(1-c)ab} \\
&= \frac{(c/a)_{n-1}(c/b)_{n-1}}{(cq)_{n-1}(c/ab)_{n-2}} cq^{n-1} \frac{(1-a)(1-b)}{(1-c)ab(1-q^{n-1}c/ab)(1-q^{n-2}c/ab)} \\
&= \frac{(c/a)_{n-1}(c/b)_{n-1}}{(cq)_{n-1}(c/abq)_{n-1}} cq^{n-1} \frac{(1-a)(1-b)(1-c/abq)}{(1-c)ab(1-q^{n-1}c/ab)(1-q^{n-2}c/ab)} \\
&= \frac{(c/a)_{n-1}(c/b)_{n-1}}{(cq)_{n-1}(c/abq)_{n-1}} \frac{(1-a)(1-b)}{(1-c)} \frac{(1-c/abq)}{(1-q^{n-1}c/ab)(1-q^{n-2}c/ab)} \frac{cq^{n-1}}{ab} \\
&= \frac{(c/a)_{n-1}(c/b)_{n-1}}{(cq)_{n-1}(c/abq)_{n-1}} \frac{(1-a)(1-b)}{(1-c)} \frac{c/abq(abq/c-1)}{q^{n-1}c/ab(q^{1-n}ab/c-1)q^{n-2}c/ab(q^{2-n}ab/c-1)} \frac{cq^{n-1}}{ab} \\
&= \frac{(c/a)_{n-1}(c/b)_{n-1}}{(cq)_{n-1}(c/abq)_{n-1}} \frac{(1-a)(1-b)}{(1-c)} \frac{(abq/c-1)}{(1-q^{1-n}ab/c)(1-q^{2-n}ab/c)} q^{-n+1} \\
&= \frac{(1-a)(1-b)q^{-n+1}(qab/c-1)}{(1-c)(1-q^{1-n}ab/c)(1-q^{2-n}ab/c)} T_{n-1}(aq, bq, cq).
\end{aligned}$$

This gives the same recurrence as for S_n. Thus, since S_n and T_n satisfy the same initial condition and recurrence relation, they must be equal for all values of n by induction. □

2.10 Lemma. Let $\pi_{k,r}(n;q)$ denote the generating function for plane partitions with at most r columns, at most k rows, and with each entry $\leq n$. Then,

$$\pi_{k,r}(n;q) = \frac{(q)_1(q)_2\cdots(q)_{k-1}}{(q)_r(q)_{r+1}\cdots(q)_{r+k-1}} \cdot \frac{(q)_{n+r}(q)_{n+r+1}\cdots(q)_{n+r+k-1}}{(q)_n(q)_{n+1}\cdots(q)_{n+k-1}}.$$

Proof. First note that $\pi_{k,r}(n;q)$ can be expressed in terms of $\pi_r(n_1,\ldots,n_k;q)$ by summing over all possible k-tuples $(n_1,\ldots,n_k)$. This simply means splitting $\pi_{k,r}(n;q)$ into a sum where each term accounts for a specific first column. Explicitly,

$$\pi_{k,r}(n;q) = \sum_{n_1=0}^{n}\sum_{n_2=0}^{n_1}\cdots\sum_{n_k=0}^{n_{k-1}} \pi_r(n_1,\ldots,n_k;q).$$

Now compare this to the generating function for plane partitions with at most $r+1$ columns, at most k rows, with each entry $\leq n$, and with the new first column consisting entirely of ns; that is, consider $\pi_{r+1}(n,n,\ldots,n;q)$. Clearly there is a one-to-one correspondence between the partitions generated by $\pi_{k,r}(n;q)$ and those generated by $\pi_{r+1}(n,n,\ldots,n;q)$; however, if N is the number being partitioned for $\pi_{k,r}(n;q)$, then $N+kn$ is the number being partitioned by $\pi_{r+1}(n,n,\ldots,n;q)$. In other words, the coefficient of q^N in $\pi_{k,r}(n;q)$ is equal to the coefficient of q^{N+kn} in $\pi_{r+1}(n,n,\ldots,n;q)$. Thus,

$$\pi_{k,r}(n;q) = q^{-kn}\pi_{r+1}(n,n,\ldots,n;q),$$

and so Lemma 2.6 can be invoked with n in place of n_j and $r+1$ in place of r, giving

$$\pi_{k,r}(n;q) = \det\left(q^{(i-j)(i-j-1)/2}\begin{bmatrix} n+r \\ r-i+j \end{bmatrix}\right)_{1\leq i,j\leq k}.$$

At this point in the book *The Theory of Partitions* [1], we read "We proceed by using another ingenious device of L. Carlitz. Let

$$W(k,r) = \det\left(q^{ri+\frac{1}{2}i(i-1)}\begin{bmatrix} j \\ i \end{bmatrix}\right)_{0\leq i,j\leq k-1}."$$

Then one multiplies $\pi_{k,r}(n;q)W(k,r)$, and the resulting determinant has a nice computable form. However, how one would come up with this $W(k,r)$ is not at all clear. Instead, following the advice of G. E. Andrews, we will define

$$V(k,r,n) = \det\left((-q)^{j-i}\begin{bmatrix} j-1 \\ j-i \end{bmatrix}\frac{(q^n;q)_{j-i}}{(q^{r+i};q)_{j-i}}\right).$$

This $V(k,r,n)$ can be found by supposing a matrix of the form

$$V(k,r,n) = \det\begin{pmatrix} 1 & X & Y & \cdots \\ 0 & 1 & Z & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $X,Y,Z,\ldots$ are chosen such that the matrix in $\pi_{k,r}(n;q)V(k,r,n)$ is upper triangular. This way, multiplying $\pi_{k,r}(n;q)$ by $V(k,r,n)$ does not affect the value of the determinant, since clearly $V(k,r,n)=1$, and we know ahead of time that the resulting matrix has a nice computable form.

So, let us multiply. If $\pi_{k,r}(n;q)V(k,r,n) = (x_{i,j})_{1\le i,j\le k}$, then

$$x_{i,j} = \sum_{t=1}^{k} q^{\binom{i-t}{2}} \begin{bmatrix} n+r \\ r-i+t \end{bmatrix} (-q)^{j-t} \begin{bmatrix} j-1 \\ j-t \end{bmatrix} \frac{(q^n)_{j-t}}{(q^{r+t})_{j-t}}.$$

Note that the factor of $\left[\begin{smallmatrix} j-1 \\ j-t \end{smallmatrix}\right]$ causes every term beyond $t = j$ to be equal to 0. Thus, the upper limit of summation can be changed to j:

$$= \sum_{t=1}^{j} q^{\binom{i-t}{2}} \begin{bmatrix} n+r \\ r-i+t \end{bmatrix} (-q)^{j-t} \begin{bmatrix} j-1 \\ j-t \end{bmatrix} \frac{(q^n)_{j-t}}{(q^{r+t})_{j-t}}.$$

Next, we should shift the index of summation so that it starts at $t = 0$ like a q-hypergeometric series:

$$= \sum_{t=0}^{j-1} q^{\binom{i-t-1}{2}} \begin{bmatrix} n+r \\ r-i+t+1 \end{bmatrix} (-q)^{j-t-1} \begin{bmatrix} j-1 \\ j-t-1 \end{bmatrix} \frac{(q^n)_{j-t-1}}{(q^{r+t+1})_{j-t-1}}.$$

Now expand the q-binomial coefficients:

$$= \sum_{t=0}^{j-1} q^{\binom{i-t-1}{2}} \frac{(q)_{n+r}}{(q)_{r-i+t+1}(q)_{n+i-t-1}} (-q)^{j-t-1} \frac{(q)_{j-1}}{(q)_{j-t-1}(q)_t} \frac{(q^n)_{j-t-1}}{(q^{r+t+1})_{j-t-1}}.$$

Next, we can split $(q)_{r-i+t+1}$ into two products:

$$= \sum_{t=0}^{j-1} q^{\binom{i-t-1}{2}} \frac{(q)_{n+r}}{(q)_{r-i+1}(q^{r-i+2})_t(q)_{n+i-t-1}} (-q)^{j-t-1} \frac{(q)_{j-1}}{(q)_{j-t-1}(q)_t} \frac{(q^n)_{j-t-1}}{(q^{r+t+1})_{j-t-1}}.$$

Also, $(q^{r+t+1})_{j-t-1}$ can be completed so that it starts with $(1-q)$:

$$= \sum_{t=0}^{j-1} q^{\binom{i-t-1}{2}} \frac{(q)_{n+r}}{(q)_{r-i+1}(q^{r-i+2})_t(q)_{n+i-t-1}} (-q)^{j-t-1} \frac{(q)_{j-1}}{(q)_{j-t-1}(q)_t} \frac{(q^n)_{j-t-1}(q)_{r+t}}{(q)_{r+j-1}}.$$

The resulting $(q)_{r+t}$ can be split into two products:

$$= \sum_{t=0}^{j-1} q^{\binom{i-t-1}{2}} \frac{(q)_{n+r}}{(q)_{r-i+1}(q^{r-i+2})_t(q)_{n+i-t-1}} (-q)^{j-t-1} \frac{(q)_{j-1}}{(q)_{j-t-1}(q)_t} \frac{(q^n)_{j-t-1}(q)_r(q^{r+1})_t}{(q)_{r+j-1}}.$$

Some products can be pulled out of the summation:

$$= \frac{(q)_{n+r}(q)_{j-1}(q)_r}{(q)_{r-i+1}(q)_{r+j-1}} \sum_{t=0}^{j-1} \frac{(q^{r+1})_t}{(q)_t(q^{r-i+2})_t} \frac{(q^n)_{j-t-1}}{(q)_{n+i-t-1}(q)_{j-t-1}} q^{\binom{i-t-1}{2}} (-q)^{j-t-1}.$$

Lemma 2.7 can be applied to each product in $(q^n)_{j-t-1}/(q)_{n+i-t-1}(q)_{j-t-1}$:

$$= \frac{(q)_{n+r}(q)_{j-1}(q)_r}{(q)_{r-i+1}(q)_{r+j-1}} \sum_{t=0}^{j-1} \frac{(q^{r+1})_t}{(q)_t(q^{r-i+2})_t} \frac{(q^n)_{j-1}}{(-1)^t q^{nt} q^{(j-1)t-\binom{t+1}{2}}(q^{2-j-n})_t}$$
$$\cdot \frac{(-1)^t q^t q^{(n+i-1)t-\binom{t+1}{2}}(q^{1-n-i})_t}{(q)_{n+i-1}}$$
$$\cdot \frac{(-1)^t q^t q^{(j-1)t-\binom{t+1}{2}}(q^{1-j})_t}{(q)_{j-1}} q^{\binom{i-t-1}{2}} (-q)^{j-t-1}.$$

Some final clean up:

$$= (-1)^{j-1} q^{j+i(i-3)/2} \frac{(q)_{n+r}(q)_r(q^n)_{j-1}}{(q)_{r-i+1}(q)_{r+j-1}(q)_{n+i-1}} \sum_{t=0}^{j-1} \frac{(q^{r+1})_t (q^{1-n-i})_t (q^{1-j})_t}{(q)_t (q^{r-i+2})_t (q^{2-j-n})_t} q^t.$$

Note that because of the factor $(q^{1-j})_t$, the upper limit of summation can be extended to ∞, and every new term is equal to zero. Thus,

$$= (-1)^{j-1} q^{j+i(i-3)/2} \frac{(q)_{n+r}(q)_r(q^n)_{j-1}}{(q)_{r-i+1}(q)_{r+j-1}(q)_{n+i-1}} \sum_{t=0}^{\infty} \frac{(q^{r+1})_t (q^{1-n-i})_t (q^{1-j})_t}{(q)_t (q^{r-i+2})_t (q^{2-j-n})_t} q^t$$

$$= (-1)^{j-1} q^{j+i(i-3)/2} \frac{(q)_{n+r}(q)_r(q^n)_{j-1}}{(q)_{r-i+1}(q)_{r+j-1}(q)_{n+i-1}} {}_3\phi_2 \left(\begin{matrix} q^{r+1} & q^{1-n-i} & q^{1-j} \\ & q^{r-i+2} & q^{2-j-n} \end{matrix} ; q, q \right).$$

Apply Lemma 2.9 to the q-hypergeometric series with a, b, c, and n respectively replaced by q^{r+1}, q^{1-n-i}, q^{r-i+2}, and $j-1$:

$$= (-1)^{j-1} q^{j+i(i-3)/2} \frac{(q)_{n+r}(q)_r(q^n)_{j-1}}{(q)_{r-i+1}(q)_{r+j-1}(q)_{n+i-1}} \frac{(q^{1-i})_{j-1}(q^{r+n+1})_{j-1}}{(q^{r-i+2})_{j-1}(q^n)_{j-1}}.$$

Cancel and combine the products that we can:

$$= (-1)^{j-1} q^{j+i(i-3)/2} \frac{(q)_r (q^{1-i})_{j-1} (q)_{n+r+j-1}}{(q)_{r+j-1}(q)_{n+i-1}(q)_{r-i+j}}.$$

Note that when $j \geq i+1$, we have $x_{i,j} = 0$ because $(q^{1-i})_{j-1}$ then contains a factor of $(1 - q^{1-i+i-1}) = (1-q^0) = 0$. Thus, the matrix $(x_{i,j})$ is lower triangular as planned, and its determinant is the product along the diagonal. To that end, compute

$$x_{i,i} = (-1)^{i-1} q^{i+i(i-3)/2} \frac{(q)_r (q^{1-i})_{i-1} (q)_{n+r+i-1}}{(q)_{r+i-1}(q)_{n+i-1}(q)_r}$$

$$= (-1)^{i-1} q^{\binom{i}{2}} \frac{(q^{1-i})_{i-1}(q)_{n+r+i-1}}{(q)_{r+i-1}(q)_{n+i-1}}$$

$$= (-1)^{i-1} q^{(i-1)+(i-2)+\cdots+1} (1-q^{1-i})(1-q^{2-i}) \cdots (1-q^{-1}) \frac{(q)_{n+r+i-1}}{(q)_{r+i-1}(q)_{n+i-1}}$$

$$= (-1)^{i-1} q^{i-1}(1-q^{1-i}) q^{2-i}(1-q^{2-i}) \cdots q^1 (1-q^{-1}) \frac{(q)_{n+r+i-1}}{(q)_{r+i-1}(q)_{n+i-1}}$$

$$= (-1)^{i-1}(q^{i-1}-1)(q^{i-2}-1)\cdots(q-1) \frac{(q)_{n+r+i-1}}{(q)_{r+i-1}(q)_{n+i-1}}$$

$$= (-1)^{i-1}(-1)^{i-1}(1-q^{i-1})(1-q^{i-2})\cdots(1-q) \frac{(q)_{n+r+i-1}}{(q)_{r+i-1}(q)_{n+i-1}}$$

$$= \frac{(q)_{i-1}(q)_{n+r+i-1}}{(q)_{r+i-1}(q)_{n+i-1}}$$

Thus,

$$\begin{aligned}
\pi_{k,r}(n;q) &= \det(x_{i,j}) \\
&= \prod_{i=1}^{k} x_{i,i} \\
&= \prod_{i=1}^{k} \frac{(q)_{i-1}(q)_{n+r+i-1}}{(q)_{r+i-1}(q)_{n+i-1}} \\
&= \frac{(q)_1(q)_2\cdots(q)_{k-1}}{(q)_r(q)_{r+1}\cdots(q)_{r+k-1}} \cdot \frac{(q)_{n+r}(q)_{n+r+1}\cdots(q)_{n+r+k-1}}{(q)_n(q)_{n+1}\cdots(q)_{n+k-1}}.
\end{aligned}$$

□

Proof of Theorem 2.1. In Lemma 2.10,

$$\pi_{k,r}(n;q) = \frac{(q)_1(q)_2\cdots(q)_{k-1}}{(q)_r(q)_{r+1}\cdots(q)_{r+k-1}} \cdot \frac{(q)_{n+r}(q)_{n+r+1}\cdots(q)_{n+r+k-1}}{(q)_n(q)_{n+1}\cdots(q)_{n+k-1}},$$

make cancellations, leaving

$$= \frac{1}{(q)_r(q^2)_{r+1}\cdots(q^k)_{r+k-1}} \cdot \frac{(q^{n+1})_r(q^{n+2})_r\cdots(q^{n+k})_r}{1},$$

send $n \to \infty$, leaving

$$\begin{aligned}
&= \frac{1}{(q)_r(q^2)_{r+1}\cdots(q^k)_{r+k-1}} \cdot \frac{(0)_r(0)_r\cdots(0)_r}{1}, \\
&= \frac{1}{(q)_r(q^2)_{r+1}\cdots(q^k)_{r+k-1}},
\end{aligned}$$

send $r \to \infty$, leaving

$$\begin{aligned}
&= \frac{1}{(q)_\infty(q^2)_\infty\cdots(q^k)_\infty}, \\
&= \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^{\min(m,k)}},
\end{aligned}$$

and send $k \to \infty$, leaving

$$= \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m}.$$

Thus, the generating function for plane partitions with at most ∞ columns, at most ∞ rows, and with each entry $\leq \infty$, i.e. $\sum_{n=0}^{\infty} p_2(n)q^n$, has the desired form. □

3 Higher Dimensional Partitions, Part 1

A d-dimensional partition of n is a d-dimensional array of positive integers whose sum is n, such that the array is non-increasing in each direction; here, the number of d-dimensional partitions of n is denoted $p_d(n)$. Note that 1- and 2-dimensional partitions are respectively the same as ordinary partitions and plane partitions; also, 3-dimensional partitions are sometimes called solid partitions. For example, $p_3(4) = 26$ because there are twenty-six solid partitions of 4, which are shown below. A subscript up arrow indicates the number should be thought of as coming out of the page, on top of the number below it; a subscript left arrow indicates the same thing, except on top of the number to the right.

$$\begin{pmatrix} 4 \end{pmatrix},$$

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1_\uparrow \\ 3 \end{pmatrix},$$

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2_\uparrow \\ 2 \end{pmatrix},$$

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1_\uparrow \\ 1_\uparrow \\ 2 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 1 \\ 1 & \end{pmatrix}, \quad \begin{pmatrix} 1_\uparrow & \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1_\uparrow \\ 2 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1_\uparrow \\ 1_\uparrow \\ 1_\uparrow \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 \\ 1 & \\ 1 & \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & & \end{pmatrix}, \quad \begin{pmatrix} 1_\uparrow & & \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1_\uparrow \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1_\uparrow & \\ 1_\uparrow & \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1_\uparrow \\ 1_\uparrow \\ 1 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1_\uparrow & 1_\uparrow \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1_\leftarrow & 1 \\ 1_\leftarrow & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1_\uparrow & \\ 1 & 1 \\ 1 & \end{pmatrix}.$$

For each dimension $d \leq 8$, the sequence of values taken on by $p_d(n)$ is available in N. J. A. Sloane's On-Line Encyclopedia of Integer Sequences (OEIS) [10]; the identifying labels for these sequences are given in Table 1. Farther down, Table 2 gives the first ten entries in each sequence.

d	OEIS
1	A000041
2	A000219
3	A000293
4	A000334
5	A000390
6	A000416
7	A000427
8	A179855

Table 1: OEIS identifying labels for the sequences of values taken on by $p_d(n)$.

The generating function for d-dimensional partitions is defined as the formal power series $\sum_{n=0}^{\infty} p_d(n)q^n$. So the question arises, does there exist an infinite product representation for dimensions higher than 2? Generalizing Theorems 1.1 and 2.1, MacMahon made the following

3.1 Conjecture. Formally,

$$\sum_{n=0}^{\infty} p_d(n)q^n \overset{?}{=} \prod_{m=1}^{\infty} (1-q^m)^{-\binom{m+d-2}{d-1}}.$$

This certainly is true for $d = 1$ and $d = 2$. Unfortunately, for all $d \geq 3$, at least when $n = 6$, the conjecture was shown to be false by A. O. L. Atkin, P. Bratley, I. G. Macdonald, and J. K. S. McKay in their 1967 paper "Some Computations for m-Dimensional Partitions" [2]. Thus, it makes sense to refer to the coefficients of q^n generated by the above infinite product as MacMahon numbers and to denote them $m_d(n)$.

3.2 Definition. Here, for all positive integers d and all nonnegative integers n, the MacMahon numbers $m_d(n)$ are defined by the equation

$$\sum_{n=0}^{\infty} m_d(n)q^n = \prod_{m=1}^{\infty} (1-q^m)^{-\binom{m+d-2}{d-1}}.$$

Table 3 gives the first ten values of $m_d(n)$ for each dimension ≤ 8, and Table 4 gives the difference $\Delta_d(n) = m_d(n) - p_d(n)$. That is one way to measure the accuracy of MacMahon's conjecture; in particular, the first nonzero difference is $\Delta_3(6) = 1$. It is possible that MacMahon actually computed this specific difference by hand, since G. E. Andrews says in his 1976 book *The Theory of Partitions* [1] that MacMahon eventually came to doubt the truth of his conjecture (p. 189).

Another way to measure the accuracy of MacMahon's conjecture is to fix n and compute $\Delta_d(n)$ as a function of d. In fact, for fixed n, both $p_d(n)$ and $m_d(n)$ seem to be polynomials with integer coefficients in the binomial basis $(1, d, \binom{d}{2}, \binom{d}{3}, \dots)$. The coefficients of $p_d(n)$ are given in Table 5; those of $m_d(n)$ in Table 6; and those of the difference $\Delta_d(n) = m_d(n) - p_d(n)$ in Table 7. The data in Table 5 came from S. B. Ekhad's 2012 paper "The Number of m-Dimensional Partitions of Eleven and Twelve" [5].

Atkin, et al. [2] found these coefficients simply by computing more points than the degree of the polynomial. Instead of that, we present a more interesting approach from *The Theory of Partitions* [1]. It is still somewhat computational, and so we only prove

3.3 Theorem. For all dimensions d,

$$m_d(6) = p_d(6) + \binom{d}{3} + \binom{d}{4}.$$

The proof of this theorem is split into Lemmas 3.4 and 3.5, which give the explicit values of $m_d(6)$ and $p_d(6)$, respectively. Thus, it suffices to simply compare the results of the two lemmas.

n	$p_1(n)$	$p_2(n)$	$p_3(n)$	$p_4(n)$	$p_5(n)$	$p_6(n)$	$p_7(n)$	$p_8(n)$
1	1	1	1	1	1	1	1	1
2	2	3	4	5	6	7	8	9
3	3	6	10	15	21	28	36	45
4	5	13	26	45	71	105	148	201
5	7	24	59	120	216	357	554	819
6	11	48	140	326	657	1197	2024	3231
7	15	86	307	835	1907	3857	7134	12321
8	22	160	684	2145	5507	12300	24796	46209
9	30	282	1464	5345	15522	38430	84625	170370
10	42	500	3122	13220	43352	118874	285784	621316
11	56	859	6500	32068	119140	362670	953430	2240838
12	77	1479	13426	76965	323946	1095430	3151332	8011584

Table 2: Values of $p_d(n)$.

n	$m_1(n)$	$m_2(n)$	$m_3(n)$	$m_4(n)$	$m_5(n)$	$m_6(n)$	$m_7(n)$	$m_8(n)$
1	1	1	1	1	1	1	1	1
2	2	3	4	5	6	7	8	9
3	3	6	10	15	21	28	36	45
4	5	13	26	45	71	105	148	201
5	7	24	59	120	216	357	554	819
6	11	48	141	331	672	1232	2094	3357
7	15	86	310	855	1982	4067	7624	13329
8	22	160	692	2214	5817	13301	27428	52215
9	30	282	1483	5545	16582	42357	96231	199686
10	42	500	3162	13741	46633	132845	332159	750733
11	56	859	6583	33362	128704	409262	1126792	2774793
12	77	1479	13602	80091	350665	1243767	3769418	10112184

Table 3: The MacMahon numbers $m_d(n)$.

n	$\Delta_1(n)$	$\Delta_2(n)$	$\Delta_3(n)$	$\Delta_4(n)$	$\Delta_5(n)$	$\Delta_6(n)$	$\Delta_7(n)$	$\Delta_8(n)$
1								
2								
3								
4								
5								
6			1	5	15	35	70	126
7			3	20	75	210	490	1008
8			8	69	310	1001	2632	6006
9			19	200	1060	3927	11606	29316
10			40	521	3281	13971	46375	129417
11			83	1294	9564	46592	173362	533955
12			176	3126	26719	148337	618086	2100600

Table 4: The difference $\Delta_d(n) = m_d(n) - p_d(n)$. Zeros are not shown.

n	1	d	$\binom{d}{2}$	$\binom{d}{3}$	$\binom{d}{4}$	$\binom{d}{5}$	$\binom{d}{6}$	$\binom{d}{7}$	$\binom{d}{8}$	$\binom{d}{9}$	$\binom{d}{10}$	$\binom{d}{11}$
1	1											
2	1	1										
3	1	2	1									
4	1	4	4	1								
5	1	6	11	7	1							
6	1	10	27	28	11	1						
7	1	14	57	93	64	16	1					
8	1	21	117	269	282	131	22	1				
9	1	29	223	707	1062	766	244	29	1			
10	1	41	417	1747	3565	3681	1871	421	37	1		
11	1	55	748	4090	10999	15489	11400	4152	683	46	1	
12	1	76	1326	9219	31828	58975	59433	31802	8483	1054	56	1

Table 5: The coefficients of the polynomials $p_d(n)$ in the binomial basis.

n	1	d	$\binom{d}{2}$	$\binom{d}{3}$	$\binom{d}{4}$	$\binom{d}{5}$	$\binom{d}{6}$	$\binom{d}{7}$	$\binom{d}{8}$	$\binom{d}{9}$	$\binom{d}{10}$	$\binom{d}{11}$
1	1											
2	1	1										
3	1	2	1									
4	1	4	4	1								
5	1	6	11	7	1							
6	1	10	27	29	12	1						
7	1	14	57	96	72	21	1					
8	1	21	117	277	319	176	38	1				
9	1	29	223	726	1186	1016	431	71	1			
10	1	41	417	1787	3926	4757	3171	1065	136	1		
11	1	55	748	4173	11961	19413	18358	9829	2666	265	1	
12	1	76	1326	9395	34250	71824	90826	69378	30531	6782	522	1

Table 6: The coefficients of the polynomials $m_d(n)$ in the binomial basis.

n	1	d	$\binom{d}{2}$	$\binom{d}{3}$	$\binom{d}{4}$	$\binom{d}{5}$	$\binom{d}{6}$	$\binom{d}{7}$	$\binom{d}{8}$	$\binom{d}{9}$	$\binom{d}{10}$	$\binom{d}{11}$
1												
2												
3												
4												
5												
6				1	1							
7				3	8	5						
8				8	37	45	16					
9				19	124	250	187	42				
10				40	361	1076	1300	644	99			
11				83	962	3924	6958	5677	1983	219		
12				176	2422	12849	31393	37576	22048	5728	466	

Table 7: The coefficients of the polynomials $\Delta_d(n) = m_d(n) - p_d(n)$ in the binomial basis.

3.4 Lemma. For all nonnegative integers d,

$$m_d(6) = 1 + 10d + 27\binom{d}{2} + 29\binom{d}{3} + 12\binom{d}{4} + \binom{d}{5}.$$

Proof. By definition, $m_d(6)$ is the coefficient of q^6 in a certain infinite product. Thus, take the infinite product, and write it out explicitly as far as there are powers of q that are ≤ 6

$$\begin{aligned}
\prod_{n=1}^{\infty}(1-q^n)^{-\binom{n+d-2}{d-1}} &= \prod_{n=1}^{\infty}\left(\frac{1}{1-q^n}\right)^{\binom{n+d-2}{d-1}} \\
&= \prod_{n=1}^{\infty}\left(\sum_{j=0}^{\infty} q^{jn}\right)^{\binom{n+d-2}{d-1}} \\
&= \left(\sum_{j=0}^{\infty} q^{j}\right)^{\binom{d-1}{d-1}} \left(\sum_{j=0}^{\infty} q^{2j}\right)^{\binom{d}{d-1}} \left(\sum_{j=0}^{\infty} q^{3j}\right)^{\binom{d+1}{d-1}} \\
&\quad \cdot \left(\sum_{j=0}^{\infty} q^{4j}\right)^{\binom{d+2}{d-1}} \left(\sum_{j=0}^{\infty} q^{5j}\right)^{\binom{d+3}{d-1}} \left(\sum_{j=0}^{\infty} q^{6j}\right)^{\binom{d+4}{d-1}} \cdots \\
&= \left(1+q+q^2+q^3+q^4+q^5+q^6+\cdots\right) \\
&\quad \cdot \left(1+q^2+q^4+q^6+\cdots\right)^{d} \left(1+q^3+q^6+\cdots\right)^{\binom{d+1}{2}} \\
&\quad \cdot \left(1+q^4+\cdots\right)^{\binom{d+2}{3}} \left(1+q^5+\cdots\right)^{\binom{d+3}{4}} \left(1+q^6+\cdots\right)^{\binom{d+4}{5}} \cdots .
\end{aligned}$$

Now use the binomial and multinomial theorems:

$$
\begin{aligned}
&= \left(1+q+q^2+q^3+q^4+q^5+q^6+\cdots\right) \\
&\quad \cdot \left(1+dq^2+\binom{d}{2}q^{2+2}+\binom{d}{3}q^{2+2+2}+dq^4+\binom{d}{1,1}q^{4+2}+dq^6+\cdots\right) \\
&\quad \cdot \left(1+\binom{d+1}{2}q^3+\binom{\binom{d+1}{2}}{2}q^{3+3}+\binom{d+1}{2}q^6+\cdots\right) \\
&\quad \cdot \left(1+\binom{d+2}{3}q^4+\cdots\right)\left(1+\binom{d+3}{4}q^5+\cdots\right)\left(1+\binom{d+4}{5}q^6+\cdots\right)\cdots \\
&= \left(1+q+q^2+q^3+q^4+q^5+q^6+\cdots\right) \\
&\quad \cdot \left(1+dq^2+\left(\binom{d}{2}+d\right)q^4+\left(\binom{d}{3}+\binom{d}{1,1}+d\right)q^6+\cdots\right) \\
&\quad \cdot \left(1+\binom{d+1}{2}q^3+\left(\binom{\binom{d+1}{2}}{2}+\binom{d+1}{2}\right)q^6+\cdots\right) \\
&\quad \cdot \left(1+\binom{d+2}{3}q^4+\cdots\right)\left(1+\binom{d+3}{4}q^5+\cdots\right)\left(1+\binom{d+4}{5}q^6+\cdots\right)\cdots .
\end{aligned}
$$

At this point, it is possible to extract the coefficient of q^6:

$$
\begin{aligned}
m_d(6) &= 1+\left(\binom{d}{3}+\binom{d}{1,1}+d\right)+\left(\binom{\binom{d+1}{2}}{2}+\binom{d+1}{2}\right)+\binom{d+4}{5}+\binom{d+3}{4} \\
&\quad +d\binom{d+2}{3}+\binom{d+2}{3}+\left(\binom{d}{2}+d\right)+d+\binom{d+1}{2}+d\binom{d+1}{2} \\
&= 1+3d+3\binom{d}{2}+\binom{d}{3}+\binom{\binom{d+1}{2}}{2}+\binom{d+4}{5}+\binom{d+3}{4} \\
&\quad +(d+1)\binom{d+2}{3}+(d+2)\binom{d+1}{2}.
\end{aligned}
$$

Note $\binom{d}{1,1}=2\binom{d}{2}$. Now use Pascal's rule $\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}$:

$$
\begin{aligned}
&= 1+3d+3\binom{d}{2}+\binom{d}{3}+\binom{\binom{d+1}{2}}{2}+\binom{d+3}{4}+\binom{d+3}{5}+\binom{d+2}{3}+\binom{d+2}{4} \\
&\quad +(d+1)\left(\binom{d+1}{2}+\binom{d+1}{3}\right)+(d+2)\left(d+\binom{d}{2}\right).
\end{aligned}
$$

Use Pascal's rule again, and pull out the part of the last term coming from the 2:

$$= 1 + 5d + 5\binom{d}{2} + \binom{d}{3} + \binom{\binom{d+1}{2}}{2} + \binom{d+2}{3} + \binom{d+2}{4} + \binom{d+2}{4} + \binom{d+2}{5}$$
$$+ \binom{d+1}{2} + \binom{d+1}{3} + \binom{d+1}{3} + \binom{d+1}{4}$$
$$+ (d+1)\left(\binom{d}{1} + \binom{d}{2} + \binom{d}{2} + \binom{d}{3}\right) + d\left(d + \binom{d}{2}\right).$$

Combine like terms:

$$= 1 + 5d + 5\binom{d}{2} + \binom{d}{3} + \binom{\binom{d+1}{2}}{2} + \binom{d+2}{3} + 2\binom{d+2}{4} + \binom{d+2}{5}$$
$$+ \binom{d+1}{2} + 2\binom{d+1}{3} + \binom{d+1}{4}$$
$$+ (d+1)\left(\binom{d}{1} + 2\binom{d}{2} + \binom{d}{3}\right) + d\left(d + \binom{d}{2}\right).$$

Use Pascal's rule again, and pull out the part of the second-to-last term coming from the 1:

$$= 1 + 6d + 7\binom{d}{2} + 2\binom{d}{3} + \binom{\binom{d+1}{2}}{2} + \binom{d+1}{2} + \binom{d+1}{3} + 2\binom{d+1}{3}$$
$$+ 2\binom{d+1}{4} + \binom{d+1}{4} + \binom{d+1}{5}$$
$$+ \binom{d}{1} + \binom{d}{2} + 2\binom{d}{2} + 2\binom{d}{3} + \binom{d}{3} + \binom{d}{4}$$
$$+ d\left(d + 2\binom{d}{2} + \binom{d}{3}\right) + d\left(d + \binom{d}{2}\right).$$

Use Pascal's rule one last time:

$$= 1 + 7d + 10\binom{d}{2} + 5\binom{d}{3} + \binom{d}{4} + \binom{\binom{d+1}{2}}{2} + \binom{d}{1} + \binom{d}{2} + \binom{d}{2} + \binom{d}{3} + 2\binom{d}{2} + 2\binom{d}{3}$$
$$+ 2\binom{d}{3} + 2\binom{d}{4} + \binom{d}{3} + \binom{d}{4} + \binom{d}{4} + \binom{d}{5}$$
$$+ d\left(2d + 3\binom{d}{2} + \binom{d}{3}\right),$$

and combine like terms:

$$= 1 + 8d + 14\binom{d}{2} + 11\binom{d}{3} + 5\binom{d}{4} + \binom{d}{5} + \binom{\binom{d+1}{2}}{2} + d\left(2d + 3\binom{d}{2} + \binom{d}{3}\right).$$

Now to deal with

$$\binom{\binom{d+1}{2}}{2} + d\left(2d + 3\binom{d}{2} + \binom{d}{3}\right),$$

first expand it in the usual way:

$$\begin{aligned}
&\binom{\binom{d+1}{2}}{2} + d\left(2d + 3\binom{d}{2} + \binom{d}{3}\right) \\
&= \frac{1}{2}\left(\frac{(d+1)d}{2}\right)\left(\frac{(d+1)d}{2} - 1\right) + d\left(2d + 3\frac{d(d-1)}{2} + \frac{d(d-1)(d-2)}{6}\right) \\
&= \frac{1}{8}\left(d^2 + d\right)\left(d^2 + d - 2\right) + \left(2d^2 + \frac{3}{2}\left(d^3 - d^2\right) + \frac{1}{6}\left(d^4 - 3d^3 + 2d^2\right)\right) \\
&= \frac{1}{8}\left(d^4 + 2d^3 - d^2 - 2d\right) + \left(\frac{1}{6}d^4 + d^3 + \frac{5}{6}d^2\right) \\
&= \frac{7}{24}d^4 + \frac{5}{4}d^3 + \frac{17}{24}d^2 - \frac{1}{4}d.
\end{aligned}$$

Next some linear algebra is in order. Specifically, the above is a vector in the vector space of polynomials in d with rational coefficients and degree ≤ 4. Right now it is represented in the standard basis $(1, d, d^2, d^3, d^4)$, and one can use the appropriate change-of-base matrix to represent it in the binomial basis $(1, d, \binom{d}{2}, \binom{d}{3}, \binom{d}{4})$. The result is

$$2d + 13\binom{d}{2} + 18\binom{d}{3} + 7\binom{d}{4},$$

which is easily verified like so:

$$\begin{aligned}
2d + 13\binom{d}{2} + 18\binom{d}{3} + 7\binom{d}{4} &= 2d + 13\frac{d(d-1)}{2} + 18\frac{d(d-1)(d-2)}{6} + 7\frac{d(d-1)(d-2)(d-3)}{24} \\
&= 2d + \frac{13}{2}\left(d^2 - d\right) + 3\left(d^3 - 3d^2 + 2d\right) + \frac{7}{24}\left(d^4 - 6d^3 + 11d^2 - 6d\right) \\
&= \frac{7}{24}d^4 + \frac{5}{4}d^3 + \frac{17}{24}d^2 - \frac{1}{4}d.
\end{aligned}$$

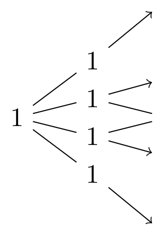
Thus,

$$\begin{aligned}
m_d(6) &= \left(1 + 8d + 14\binom{d}{2} + 11\binom{d}{3} + 5\binom{d}{4} + \binom{d}{5}\right) + \left(2d + 13\binom{d}{2} + 18\binom{d}{3} + 7\binom{d}{4}\right) \\
&= 1 + 10d + 27\binom{d}{2} + 29\binom{d}{3} + 12\binom{d}{4} + \binom{d}{5}. \qquad \square
\end{aligned}$$

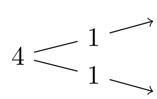
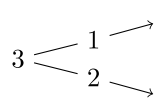
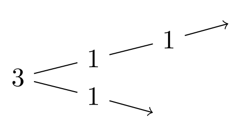
3.5 Lemma. For all nonnegative integers d,

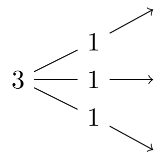
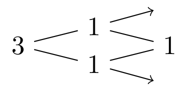
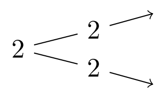
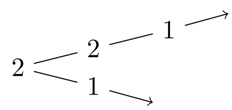
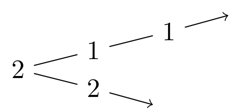
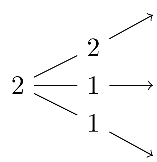
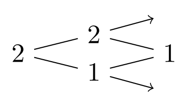
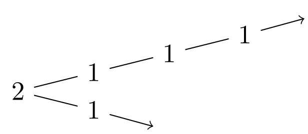
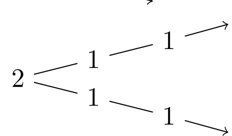
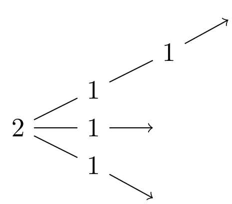
$$p_d(6) = 1 + 10d + 27\binom{d}{2} + 28\binom{d}{3} + 11\binom{d}{4} + \binom{d}{5}$$

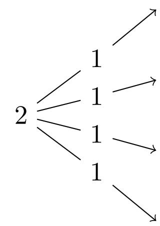
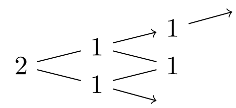
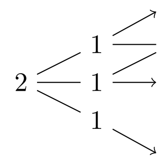
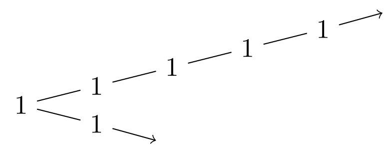
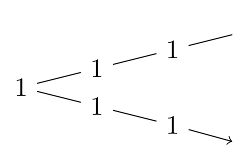
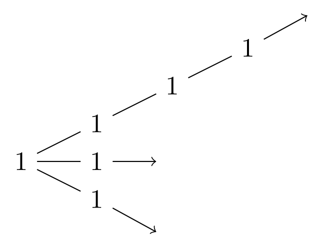
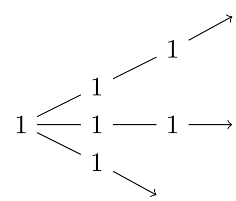
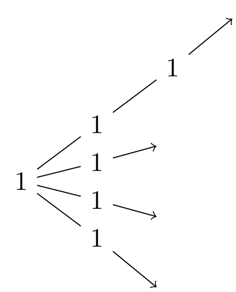
Proof. In Table 8 below, for each ordinary partition of 6, we list the ways to arrange the partition in higher dimensional space, and then for each arrangement, we count the number of ways to place it in higher dimensional space. For example, one ordinary partition of 6 is $1+1+1+1+1+1$, and one arrangement of this partition is as follows.



Each arrowhead indicates a separate dimension of the d-dimensional space; here, four dimensions are used. There is a 1 at $(0,0,0,0)$, $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, and also one at a point for which two coordinates are 1 and two coordinates are 0, such as $(1,1,0,0)$. There are $\binom{d}{4}$ ways to choose which four dimensions of the d-dimensional space are used, and for each one of those ways, there are $\binom{4}{2} = 6$ ways to choose the point of the form $(1,1,0,0)$. Thus, there are $6\binom{d}{4}$ placements of this arrangement of this partition. The value of $p_d(6)$ is given by the sum total of the right-hand column in the table.

Ordinary Partition of 6	Arrangement in Higher Dimensional Space	Placements
6	6	1
$5+1$	$5 — 1 \longrightarrow$	d
$4+2$	$4 — 2 \longrightarrow$	d
$4+1+1$	$4 — 1 — 1 \longrightarrow$	d
		$\binom{d}{2}$
$3+3$	$3 — 3 \longrightarrow$	d
$3+2+1$	$3 — 2 — 1 \longrightarrow$	d
		$2\binom{d}{2}$
$3+1+1+1$	$3 — 1 — 1 — 1 \longrightarrow$	d
		$2\binom{d}{2}$

		$\binom{d}{3}$
		$\binom{d}{2}$
$2+2+2$	$2 - 2 - 2 \longrightarrow$	d
		$\binom{d}{2}$
$2+2+1+1$	$2 - 2 - 1 - 1 \longrightarrow$	d
		$2\binom{d}{2}$
		$2\binom{d}{2}$
		$3\binom{d}{3}$
		$2\binom{d}{2}$
$2+1+1+1+1$	$2 - 1 - 1 - 1 - 1 \longrightarrow$	d
		$2\binom{d}{2}$
		$\binom{d}{2}$
		$3\binom{d}{3}$

		$\binom{d}{4}$
		$2\binom{d}{2}$
		$3\binom{d}{3}$
$1+1+1+1+1+1$	$1 — 1 — 1 — 1 — 1 — 1 \longrightarrow$	d
		$2\binom{d}{2}$
		$2\binom{d}{2}$
		$3\binom{d}{3}$
		$3\binom{d}{3}$
		$4\binom{d}{4}$

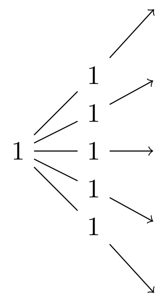
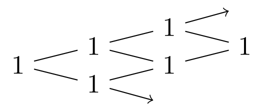
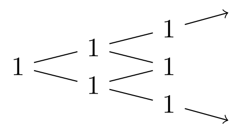
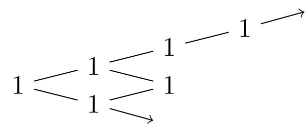
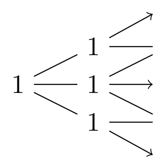
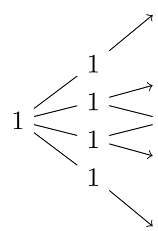
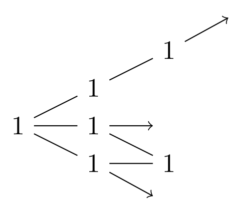
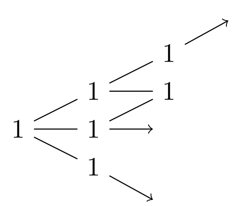
		$\binom{d}{5}$
		$2\binom{d}{2}$
		$\binom{d}{2}$
		$2\binom{d}{2}$
		$3\binom{d}{3}$
		$6\binom{d}{4}$
		$3\binom{d}{3}$
		$6\binom{d}{3}$

Table 8: Arrangements of ordinary partitions of 6 into higher dimensional space.

□

4 Higher Dimensional Partitions, Part 2

Despite the above disproof, not all hope is lost for MacMahon's conjecture; for several physicists relatively recently showed it to be asymptotically correct in a certain way. The path to this was as follows.

The familiar 2-dimensional Ferrers diagram of an ordinary partition generalizes to a $(d+1)$-dimensional Ferrers array for d-dimensional partitions. This representation of d-dimensional partitions was rediscovered by F. Y. Wu, G. Rollet, H. Y. Huang, J. M. Maillard, C.-K. Hu, and C.-N. Chen in their 1996 paper "Directed Compact Lattice Animals, Restricted Partitions of an Integer, and the Infinite-State Potts Model" [11]. They conjectured that the number of d-dimensional partitions of n, with restrictions on the sizes of the parts, is asymptotically equivalent to $\exp(Cn^{d/(d+1)})$.

This was proved by D. P. Bhatia, M. A. Prasad, and D. Arora in their 1997 paper "Asymptotic Results for the Number of Multidimensional Partitions of an Integer and Directed Compact Lattice Animals" [4], where they also proved the conjecture for partitions without restrictions on the sizes of the parts; that is, for $p_d(n)$. We state this last result as

4.1 Theorem. For every nonnegative integer d, we have that $\log p_d(n) \asymp n^{d/(d+1)}$ as $n \to \infty$; that is, there exist positive constants C_1 and C_2, depending on d, such that

$$C_1 \log n^{d/(d+1)} \le \log p_d(n) \le C_2 \log n^{d/(d+1)},$$

for sufficiently large n.

This theorem, with the MacMahon numbers $m_d(n)$ in place of $p_d(n)$, was shown to hold in Chapter 5 of N. S. Prabhakar's 2011 Bachelor's thesis "On the Asymptotics of Some Counting Problems in Physics" [9], and in a more polished form in Appendix A of the 2012 paper "On the Asymptotics of Higher Dimensional Partitions" by S. Balakrishnan, S. Govindarajan, and N. S. Prabhakar [3].

We present the proof of Bhatia, et al. [4] in Lemmas 4.2 and 4.3, the former for the lower bound and the latter for the upper bound. Both use clever elementary counting arguments, but the second also uses induction on the dimension d and Ferrers arrays.

4.2 Lemma. There exists a positive constant C, depending on d, such that

$$Cn^{d/(d+1)} \le \log p_d(n)$$

for sufficiently large n.

Proof. The idea is to explicitly describe a set of particularly simple d-dimensional partitions. First, one requires that all parts be arranged on a fixed d-dimensional cube of side length s. Not all partitions under consideration will be partitions of the same number n, but they will be partitions of numbers in an interval $[n_1, n_2]$. This gives enough information to obtain a lower bound on $p_d(n)$ because it is an increasing function of n, and so

$$\sum_{n=n_1}^{n_2} p_d(n) \le (n_2 - n_1 + 1)\, p_d(n_2). \tag{1}$$

The cube contains s^d lattice points $(x_1, \ldots, x_d)$ with $0 < x_j \le s$ for all $j \in \{1, \ldots, d\}$, and one obtains partitions by assigning a positive integer to each one of these lattice points. These should be thought of as the parts of the partition; their sum is the number being partitioned. Specifically, consider assignments of the form

$$P(x_1, \ldots, x_d) \;=\; sd - (x_1 + \cdots + x_d) + \delta,$$

where $\delta \in \{0, 1\}$ depends on the point $(x_1, \ldots, x_d)$. Because of the amount of freedom in choosing δ, there are 2^{s^d} such assignments, and therefore, the same number of partitions. Note that one cannot take $\delta > 1$ and still expect to get a partition, because then the resulting array of numbers is not necessarily non-increasing in every direction. Explicitly, the non-increasing condition requires that

$$\begin{aligned} &P(x_1, \dots, x_j + 1, \dots, x_d) - P(x_1, \dots, x_j, \dots, x_d) \\ &\qquad = -1 + \delta(x_1, \dots, x_j + 1, \dots, x_d) - \delta(x_1, \dots, x_j, \dots, x_d) \le 0. \end{aligned}$$

The bound obtained thus far is

$$2^{s^d} \le \sum_{n=n_1}^{n_2} p_d(n), \tag{2}$$

where n_1 and n_2 are respectively the smallest and largest numbers being partitioned. Of course these two numbers correspond respectively to $\delta \equiv 0$ and $\delta \equiv 1$. That is,

$$\begin{aligned} n_1 &= \sum_{x_1=1}^{s} \cdots \sum_{x_d=1}^{s} sd - (x_1 + \cdots + x_d) \\ &= s^d(sd) - \left(\sum_{x_1=1}^{s} x_1 + \cdots + \sum_{x_d=1}^{s} x_d \right) \\ &= s^{d+1}d - d\left(\frac{s(s+1)}{2}\right), \end{aligned} \tag{3}$$

and $n_2 = n_1 + s^d$. Now combining this with (1) and (2) gives

$$2^{s^d} \le (s^d + 1)\, p_d(n_2), \tag{4}$$

but what is the size of s in terms of n_2? From (3), it is clear that $n_1 \asymp s^{d+1}$ as $s \to \infty$, and since $n_2 = n_1 + s^d$, the same is true for n_2. Thinking only of those integers n_2 that arise as above for some integer s, this can be reversed to say that

$$s \asymp n_2^{1/(d+1)}$$

as $n_2 \to \infty$. Using this information in (4), taking logarithms, and then isolating the term of interest:

$$\begin{aligned} 2^{Cn_2^{d/(d+1)}} &\le (Cn_2^{d/(d+1)} + 1)\, p_d(n_2) \\ Cn_2^{d/(d+1)} &\le \log(Cn_2^{d/(d+1)} + 1) + \log(p_d(n_2)) \\ Cn_2^{d/(d+1)} - \log(Cn_2^{d/(d+1)} + 1) &\le \log(p_d(n_2)). \end{aligned}$$

Now the end is in sight, since the left-hand side is $\asymp n_2^{d/(d+1)}$. However, this only covers those integers in the sequence

$$n_2 = n_2(s) = s^{d+1}d - d\left(\frac{s(s+1)}{2}\right) + s^d.$$

If n is not in this sequence, take s to be the largest side length such that $n_2(s) < n$. Then (4) still holds, since $p_d(n_2(s)) < p_d(n)$. Furthermore, $n - n_2(s) < n_2(s+1) - n_2(s) \asymp s^d$ as $n \to \infty$, so all the asymptotics still follow with n in place of n_2. □

4.3 Lemma. For every nonnegative integer d, there exists a positive constant C, depending on d, such that

$$\log p_d(n) \leq C n^{d/(d+1)}$$

for sufficiently large n.

Proof. Induction on d. For the base case, Bhatia, et al. [4] used plane partitions, citing Wu, et al. [11]; however, that paper does not contain a rigorous proof. Instead, we use ordinary partitions for the base case, using a result in M. I. Knopp's 1970 book *Modular Functions in Analytic Number Theory* [7].

First, consider the generating function of ordinary partitions, $G(x) = \sum_{n=0}^{\infty} p_1(n)x^n$. For $0 < x < 1$, all the terms in the series are positive; therefore, the whole sum is larger than any single summand. That is, $G(x) \geq p_1(n)x^n$, or equivalently, $p_1(n) \leq G(x)/x^n$. Taking the logarithm of both sides yields $\log p_1(n) \leq \log G(x) - n \log x$.

Next, look at $\log G(x)$ using Theorem 1.1:

$$\begin{aligned}
\log G(x) &= \log\left(\prod_{m=1}^{\infty} \frac{1}{1-x^m}\right) \\
&= \sum_{m=1}^{\infty} \log\left(\frac{1}{1-x^m}\right) \\
&= \sum_{m=1}^{\infty} -\log\left(1-x^m\right) \\
&= \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \frac{(x^m)^k}{k} \\
&= \sum_{k=1}^{\infty} \frac{1}{k} \sum_{m=1}^{\infty} x^{mk} \\
&= \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{x^k}{1-x^k}\right).
\end{aligned}$$

At this point, a crafty inequality is needed. Let $f(x) = x^k$, and note that on $[0, 1]$ this function is increasing and concave up; therefore the tangent lies below the secant. That is,

$$\begin{aligned}
f'(x) &< \frac{f(1) - f(x)}{1-x} \\
&\implies kx^{k-1} < \frac{1-x^k}{1-x} \\
&\implies \frac{x^{k-1}}{1-x^k} < \frac{1}{k(1-x)} \\
&\implies \frac{x^k}{1-x^k} < \frac{x}{k(1-x)} \\
&\implies \frac{1}{k}\left(\frac{x^k}{1-x^k}\right) < \frac{x}{k^2(1-x)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\log G(x) &< \sum_{k=1}^{\infty} \frac{x}{k^2(1-x)} \\
&= \frac{x}{1-x} \sum_{k=1}^{\infty} \frac{1}{k^2} \\
&= \frac{\pi^2}{6}\left(\frac{x}{1-x}\right),
\end{aligned}$$

using the solution to the famous Basel problem.

Now to investigate $n \log x$, again with a crafty inequality. This time, let $g(x) = \log(x)$, and note that on $[1, 1/x]$ this function is increasing and concave down; therefore the function lies below the tangent, or $g(x) < x - 1$. In particular,

$$\begin{aligned}
-\log(x) &= \log(1/x) \\
&< 1/x - 1 \\
&= \frac{1-x}{x}.
\end{aligned}$$

Thus, altogether,

$$\begin{aligned}
\log p_1(n) &\leq \log G(x) - n \log x \\
&\leq \frac{\pi^2}{6}\left(\frac{x}{1-x}\right) + n\left(\frac{1-x}{x}\right).
\end{aligned}$$

Next, choose the value of x that makes the two terms equal, namely

$$x = \frac{\sqrt{6n}}{\pi + \sqrt{6n}}.$$

This yields the following:

$$\begin{aligned}
\log p_1(n) &\leq \frac{\pi^2}{6}\left(\frac{\frac{\sqrt{6n}}{\pi+\sqrt{6n}}}{1 - \frac{\sqrt{6n}}{\pi+\sqrt{6n}}}\right) + n\left(\frac{1 - \frac{\sqrt{6n}}{\pi+\sqrt{6n}}}{\frac{\sqrt{6n}}{\pi+\sqrt{6n}}}\right) \\
&= \frac{\pi^2}{6}\frac{\sqrt{6n}}{\pi} + n\frac{\pi}{\sqrt{6n}} \\
&= \frac{\pi\sqrt{6n}}{6} + \frac{\pi\sqrt{6n}}{6} \\
&= \frac{2\pi\sqrt{6n}}{6} \\
&= \frac{\pi\sqrt{6n}}{3} \\
&= \frac{\pi\sqrt{2}\sqrt{3}\sqrt{n}}{\sqrt{3}\sqrt{3}} \\
&= \pi\sqrt{\frac{2}{3}}\sqrt{n},
\end{aligned}$$

which completes the proof of the base case $d = 1$, with $C = \pi\sqrt{\frac{2}{3}}$.

For the inductive step, suppose the upper bound holds for some dimension $(d-1) \geq 1$, and fix a number n to be partitioned in d dimensions. In order to make use of the inductive hypothesis, it will be necessary to decompose a d-dimensional partition into a list of $(d-1)$-dimensional partitions. In this way, one arrives at an upper bound of the form

$$p_d(n) \leq \sum \prod_k p_{d-1}(k), \tag{5}$$

where the sum is over some set S of 1-dimensional partitions of n, and the product is over the parts k of these 1-dimensional partitions of n. Each term in the sum represents a way to partition n into a 1-dimensional list of parts k, and each factor in the product represents the number of ways each part k can be subsequently partitioned into a $(d-1)$-dimensional array. The difficult part of the argument is to make S small enough to give the claimed upper bound.

As one might expect, the set of *all* 1-dimensional partitions of n is too large. This corresponds to fixing a dimension and mindlessly listing the $(d-1)$-dimensional partitions along this fixed coordinate axis. For example, any 2-dimensional partition can be viewed as a row of 1-dimensional partitions, and any 3-dimensional partition can be viewed as a stack of 2-dimensional partitions. The longest list occurs when the d-dimensional partition is a line made entirely of 1's.

This extreme case illustrates the problem, but it also suggests a solution. If one chooses a direction other than the direction of the line of 1's, then the list of $(d-1)$-dimensional partitions will be of length 1. So rather than fixing a direction before constructing the list, start with the largest $(d-1)$-dimensional partition, whatever direction it may lie in, and then from the remaining parts, again choose the largest $(d-1)$-dimensional partition, even if it's not in the same direction, and continue in this way until all parts are accounted for. Informally, one can picture a 3-dimensional partition as a packing of walls and floors, rather than just a stack of floors.

How long can a list of $(d-1)$-dimensional partitions be when constructed in this way? To maximize the length of such a list, of course all parts of the d-dimensional partition should be 1. Then, one should minimize the size of the largest $(d-1)$-dimensional partition at each step by packing the the parts into a cube. This particular list will actually proceed along a fixed direction, simply taking $\lceil n^{1/d} \rceil$ slices of the cube of side length $\lceil n^{1/d} \rceil$. In terms of the set S of 1-dimensional partitions of n, this provides an upper bound of $\lceil n^{1/d} \rceil$ on both the size of the parts and the total number of parts. This is not sufficient to directly obtain an upper bound on $p_d(n)$ of the form $n^{d/(d+1)}$.

In order to force $(d+1)$ to appear, consider the $(d+1)$-dimensional Ferrers array of a d-dimensional partition. This can be viewed as a $(d+1)$-dimensional partition where all the parts are 1's. Thus, by the preceding discussion, this can be decomposed into a list of d-dimensional partitions, where all the parts are still 1's, and with the length of the list at most $\lceil n^{1/(d+1)} \rceil$. This of course corresponds to a list of d-dimensional Ferrers arrays, which in turn correspond to $(d-1)$-dimensional partitions. Again the upper bound $\lceil n^{1/(d+1)} \rceil$ on the length of the list is an upper bound on the number of parts in the 1-dimensional partitions in the set S. There is no longer a clear bound on the sizes of the parts of these 1-dimensional partitions, but this will not be a problem.

So what can be said about the bound (5) now? First, invoking the base case, the number of terms in the sum is bounded by $\exp(Cn^{1/2})$. Second, each product can be extended to have exactly $\lceil n^{1/(d+1)} \rceil$ factors by multiplying by 1 several times, and so the maximum value of this product occurs when all the factors are equal. That is, when all the parts are made as close as possible to

$$\frac{n}{\lceil n^{1/(d+1)} \rceil} \asymp \frac{n}{n^{1/(d+1)}} = n^{d/(d+1)}.$$

Thus, (5) becomes

$$\begin{aligned}
p_d(n) &\le \sum \prod_k p_{d-1}(k) \\
&\le \exp(Cn^{1/2}) \, \left(p_{d-1}\left(n^{d/(d+1)}\right)\right)^{\lceil n^{1/(d+1)} \rceil} \\
&\le \exp(Cn^{1/2}) \, \left(p_{d-1}\left(n^{d/(d+1)}\right)\right)^{n^{1/(d+1)}+1}.
\end{aligned}$$

Finally making use of the inductive hypothesis, for sufficiently large n, this is

$$\begin{aligned}
&\le \exp(Cn^{1/2}) \, \left(\exp\left(C\left(n^{d/(d+1)}\right)^{(d-1)/d}\right)\right)^{n^{1/(d+1)}+1} \\
&= \exp(Cn^{1/2}) \, \exp\left(C\left(n^{(d-1)/(d+1)}\right)\left(n^{1/(d+1)}+1\right)\right) \\
&= \exp(Cn^{1/2}) \, \exp\left(Cn^{d/(d+1)} + Cn^{(d-1)/(d+1)}\right).
\end{aligned}$$

Taking logarithms,

$$\begin{aligned}
\log p_d(n) &\le Cn^{1/2} + Cn^{d/(d+1)} + Cn^{(d-1)/(d+1)} \\
&\le Cn^{d/(d+1)}. \qquad \square
\end{aligned}$$

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[1]This author is really D. Zeilberger's computer.